

SMALL EXTENSIONS OF ABELIAN ORDERED GROUPS

Lecture I. Abelian ordered groups

1. ORDERED SETS

In these notes, an *ordered set* will be a set equipped with a total ordering.

Notation. Let I, J be ordered sets.

- I_∞ is the ordered set obtained by adding a (new) last element, which is formally denoted as ∞ .
- I^{opp} is the ordered set obtained by reversing the ordering of I .
- For $S, T \subset I$ and $i \in I$, the following expressions have the obvious meaning

$$i < S, \quad i > S, \quad S < T.$$

- $I + J$ is the disjoint union $I \sqcup J$ with the total ordering which respects the orderings of I and J and satisfies $I < J$.

An *initial segment* of an ordered set I is a subset $S \subset I$ such that

$$j \in I, i \in S, j < i \implies j \in S.$$

We denote by $\text{Init}(I)$ the set of initial segments of I .

Note that $\text{Init}(I)$ is an ordered set with respect to inclusion.

A mapping $\iota: I \rightarrow J$ between two ordered sets is an *embedding* if it strictly preserves the ordering

$$x < y \implies \iota(x) < \iota(y), \quad \forall x, y \in I.$$

We also say that $\iota: I \rightarrow J$ is an *extension* of I .

An *isomorphism* of ordered sets is an onto embedding. The *order-type* of an ordered set is the class of this set up to isomorphism.

For well-ordered sets the order-type is an ordinal number, like

$$\text{order-type}(I) = 7, \quad \omega, \quad \omega^3 \cdot 4 + 11,$$

where ω is the order-type of \mathbb{N} . We agree that $0 \notin \mathbb{N}$.

2. ORDERED GROUPS

An *ordered group* (Γ, \leq) is an (additive) abelian group Γ equipped with a total ordering \leq , which is compatible with the group structure:

$$\beta < \gamma \implies \beta + \rho < \gamma + \rho, \quad \forall \beta, \gamma, \rho \in \Gamma.$$

For any $\gamma \in \Gamma$, we shall denote

$$|\gamma| = \text{Max}(\gamma, -\gamma).$$

An ordered group Γ has no torsion. In fact, any non-zero $\gamma \in \Gamma$ satisfies

$$n|\gamma| > |\gamma| > 0, \quad \forall n \in \mathbb{N}.$$

An *embedding/extension/isomorphism* of ordered groups is a group homomorphism which is simultaneously an embedding/extension/isomorphism of ordered sets.

Basic examples.

- $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \pi) \subset \mathbb{R}$.
- $\mathbb{Z}_{\text{lex}}^2 \subset \mathbb{Q}_{\text{lex}}^2 \subset \mathbb{Q}(\sqrt{2}, \pi)_{\text{lex}}^2 \subset \mathbb{R}_{\text{lex}}^2$.

Hahn sum and Hahn product. Let I be an ordered set, and let $(\Gamma_i)_{i \in I}$ be a family of ordered groups parameterized by I .

We define their *Hahn sum* as the direct sum

$$\coprod_{i \in I} \Gamma_i := \bigoplus_{i \in I} \Gamma_i,$$

equipped with the lexicographical order.

Consider the product $\prod_{i \in I} \Gamma_i$. For any element $a = (a_i)_{i \in I}$ in this product, the *support* of a is the subset $\text{supp}(a) \subset I$ formed by all indices i with $a_i \neq 0$.

We define the *Hahn product*

$$\left(\prod_{i \in I} \Gamma_i \right)_{\text{lex}} \subset \prod_{i \in I} \Gamma_i$$

as the subgroup formed by all elements whose support is a well-ordered subset of I , with respect to the ordering induced by that of I .

It is easy to check that it makes sense to consider the lexicographical order on this subgroup.

Clearly, the Hahn product is an extension of the Hahn sum:

$$\coprod_{i \in I} \Gamma_i \subset \left(\prod_{i \in I} \Gamma_i \right)_{\text{lex}}.$$

If all ordered groups coincide, $\Gamma_i = \Gamma$ for all $i \in I$, then we use the notation

$$\Gamma^{(I)} \subset \Gamma_{\text{lex}}^I,$$

for the Hahn sum and product, respectively.

Ordered groups as valuation groups. Every ordered group Γ is the value group of some valued field. For any field k , we may consider the group algebra $k[\Gamma]$, whose elements may be expressed as:

$$\sum_{\gamma \in S} a_\gamma t^\gamma, \quad a_\gamma \in k,$$

where S is a finite subset of Γ and t is a formal symbol.

This ring admits the valuation

$$v: k[\Gamma] \longrightarrow \Gamma \cup \{\infty\}, \quad \sum_{\gamma \in S} a_\gamma t^\gamma \longmapsto \text{Min}\{\gamma \in S \mid a_\gamma \neq 0\}.$$

The support of the valuation is $v^{-1}(\infty) = \{0\}$. This proves that $k[\Gamma]$ is an integral domain, and the valuation may be extended to its field of fractions.

Divisible hull. A torsion-free group G is *divisible* if for all $\gamma \in G$ and all $n \in \mathbb{N}$, there exists (a necessarily unique) $\beta \in G$ such that $n\beta = \gamma$.

For any ordered group Γ the group

$$\Gamma_{\mathbb{Q}} := \Gamma \otimes \mathbb{Q}$$

is divisible, and it has a natural structure of ordered group with the ordering determined by the condition

$$\gamma \otimes (1/n) < \beta \otimes (1/m) \iff m\gamma < n\beta,$$

for all $n, m \in \mathbb{N}$ and all $\gamma, \beta \in \Gamma$.

Since Γ has no torsion, it may be embedded in a unique way into $\Gamma_{\mathbb{Q}}$ as an ordered group. We say that $\Gamma_{\mathbb{Q}}$ is the *divisible hull* of Γ , because it is the minimal divisible extension of Γ .

Lemma 2.1. *For any embedding $\iota: \Gamma \hookrightarrow \Lambda$ into a divisible ordered group Λ , there exists a unique embedding of $\Gamma_{\mathbb{Q}}$ into Λ such that ι coincides with the composition $\Gamma \hookrightarrow \Gamma_{\mathbb{Q}} \hookrightarrow \Lambda$.*

3. CONVEX SUBGROUPS AND RANK

Any subgroup $H \subset \Gamma$ inherits the structure of ordered group, just by taking the induced ordering. However, the quotient Γ/H does not always inherit a structure of ordered group.

This occurs if and only if H is a *convex* subgroup; that is,

$$\beta \in \Gamma, \gamma \in H, |\beta| < |\gamma| \implies \beta \in H.$$

In this case, we may define an ordering in Γ/H by:

$$\beta + H < \rho + H \iff \beta + H \neq \rho + H \text{ and } \beta < \rho.$$

The notation $\beta + H < \rho + H$ is compatible with the natural meaning of such an inequality for arbitrary subsets of Γ . That is, every element in $\beta + H$ is smaller than any element in $\rho + H$.

Lemma 3.1. *Let $f: \Gamma \rightarrow \Delta$ be an order-preserving group homomorphism between two ordered groups. Then, $\text{Ker}(f)$ is a convex subgroup of Γ and the natural isomorphism between $\Gamma/\text{Ker}(f)$ and $f(\Gamma)$ is order-preserving too.*

Lemma 3.2. *The convex subgroups of Γ are totally ordered by inclusion.*

Proof. Let H, H' be convex subgroups such that there exists $\gamma \in H \setminus H'$.

Then, for all $\beta \in H'$ we must have $|\beta| < |\gamma|$, so that $H' \subset H$. □

Definition. Let $\text{Cvx} = \text{Cvx}(\Gamma)$ be the ordered set of all *proper* convex subgroups, ordered by increasing inclusion

$$\{0\} \subset \dots \subset H \subset \dots$$

The order-type of Cvx is called the *rank* of Γ , and is denoted $\text{rk}(\Gamma)$.

We may identify Cvx_{∞} with the ordered set of all convex subgroups of Γ , by letting ∞ represent the whole group Γ .

Examples.

- $\text{rk}(\mathbb{Z}) = \text{rk}(\mathbb{Q}) = \text{rk}(\mathbb{R}) = 1$.
- $\text{rk}(\mathbb{R}_{\text{lex}}^n) = n$. The sequence of convex subgroups is

$$\{0\} \subset \{0\} \times \cdots \times \{0\} \times \mathbb{R} \subset \cdots \subset \{0\} \times \mathbb{R} \times \cdots \times \mathbb{R} \subset \mathbb{R}_{\text{lex}}^n,$$
- $\text{rk}(\mathbb{R}_{\text{lex}}^{\mathbb{N}}) = \text{order-type}((\mathbb{N}\infty)^{\text{opp}})$, $\text{rk}(\mathbb{R}_{\text{lex}}^{\mathbb{Q}}) > \text{order-type}((\mathbb{Q}\infty)^{\text{opp}})$.
- $\text{rk}(\Gamma) = \text{rk}(\Gamma_{\mathbb{Q}})$.

Principal convex subgroups. Let Γ be an ordered group. For any $\gamma \in \Gamma$, we denote by $H(\gamma)$ the convex subgroup of Γ generated by γ . That is,

$$H(\gamma) = \{\beta \in \Gamma \mid |\beta| \leq n|\gamma| \text{ for some } n \in \mathbb{N}\}.$$

Equivalently, $H(\gamma)$ is the intersection of all convex subgroups containing γ .

These convex subgroups $H(\gamma)$ are said to be *principal*.

Definition. Let $I = \text{PrCvx}(\Gamma)$ be the ordered set of *non-zero* convex principal subgroups of Γ , ordered by decreasing inclusion.

$$\cdots \supset H(\gamma) \supset \cdots$$

The order-type of I is called the *principal rank* of Γ , and is denoted $\text{prk}(\Gamma)$.

We may identify $I\infty$ with a set of indices parameterizing all principal convex subgroups of Γ . For any $i \in I$ we shall denote by H_i the corresponding principal convex subgroup. We agree that $H_\infty = \{0\}$.

Then, according to our convention, for any pair of indices $i, j \in I\infty$, we have

$$i < j \iff H_i \supsetneq H_j.$$

Lemma 3.3. *Every convex subgroup $H \subset \Gamma$ satisfies $H = \bigcup_{i \in I, H_i \subset H} H_i$.*

Proof. For all $\gamma \in H$, the principal convex subgroup $H(\gamma)$ is contained in H . \square

Corollary 3.4. *If I is well-ordered, then all convex subgroups are principal.*

Proof. For any convex subgroup H , the subset $\{i \in I \mid H_i \subset H\} \subset I$ has a minimal element i_0 . By Lemma 3.3, $H = H_{i_0}$. \square

Skeleton of an ordered group and immediate extensions. If $H = H(\gamma)$ is a non-zero principal convex subgroup of Γ , we denote by H^* the union of all principal convex subgroups not containing γ .

Clearly, $H \supsetneq H^*$ is a convex subgroup (not necessarily principal) and there are no convex subgroups between these two subgroups.

In other words, H^* is the immediate predecessor of H in the ordered set $\text{Cvx}\infty$. In particular, *the quotient H/H^* is an ordered group of rank one.*

Definition. This quotient H/H^* is said to be the *component* of Γ determined by the non-zero principal convex subgroup H .

The component of Γ determined by any $i \in I$ will be denoted as

$$C_i = C_i(\Gamma) = H_i/H_i^*.$$

The *skeleton* of Γ is the pair $(I, (C_i)_{i \in I})$.

Lemma 3.5. *Let $\Gamma \hookrightarrow \Lambda$ be an extension of ordered groups. The mappings*

$$\begin{aligned} \text{Cvx}(\Gamma) &\longrightarrow \text{Cvx}(\Lambda), & H &\longmapsto \text{convex subgroup of } \Lambda \text{ generated by } H, \\ \text{PrCvx}(\Gamma) &\xrightarrow{\iota} \text{PrCvx}(\Lambda), & H(\gamma) &\longmapsto \text{convex subgroup of } \Lambda \text{ generated by } \gamma, \end{aligned}$$

are embeddings of ordered sets. Thus, $\text{rk}(\Gamma) \leq \text{rk}(\Lambda)$ and $\text{prk}(\Gamma) \leq \text{prk}(\Lambda)$.

Moreover, if $i \in I = \text{PrCvx}(\Gamma)$ is the element that corresponds to $H(\gamma)$, then the embedding $H(\gamma) \subset H_\Lambda(\gamma)$ induces an embedding $C_i(\Gamma) \hookrightarrow C_{\iota(i)}(\Lambda)$ between their respective components.

Definition. The extension $\Gamma \hookrightarrow \Lambda$ is *immediate* if it preserves the skeleton. That is, it induces an isomorphism $\text{PrCvx}(\Gamma) \simeq \text{PrCvx}(\Lambda)$ of ordered sets, and isomorphisms $C_i(\Gamma) \simeq C_{\iota(i)}(\Lambda)$ between all the components.

If an extension $\Gamma \hookrightarrow \Lambda$ is immediate, then $\text{prk}(\Gamma) = \text{prk}(\Lambda)$ by the very definition of the principal rank. Also, one has $\text{rk}(\Gamma) = \text{rk}(\Lambda)$ by Lemma 3.7.

The converse implication is not true. All non-trivial subgroups of \mathbb{R} have rank one, but they have many different skeletons.

Ordered groups with a prefixed skeleton. Let I be an ordered set and $(C_i)_{i \in I}$ a family of ordered groups of rank one, parameterized by I .

The Hahn sum and product

$$\prod_{i \in I} C_i \subset \left(\prod_{i \in I} C_i \right)_{\text{lex}}$$

have both skeleton $(I, (C_i)_{i \in I})$.

More precisely, let Γ denote any one of these two groups, the Hahn sum or the Hahn product. For each $i \in I$, consider the following subgroup of Γ :

$$H_i = \{(a_j)_{j \in I} \mid a_j = 0 \text{ for all } j < i\}.$$

Then, H_i is the principal subgroup of Γ generated by any $(a_j)_{j \in I} \in H_i$ with $a_i \neq 0$.

Also, the assignment $i \mapsto H_i$ determines an isomorphism of ordered sets between I and $I(\Gamma)$. In particular, $\text{prk}(\Gamma)$ is the order-type of I .

Moreover, the projection homomorphism

$$H_i \longrightarrow C_i, \quad (a_j)_{j \in I} \longmapsto a_i$$

induces an isomorphism of ordered groups between H_i/H_i^* and C_i .

Relationship between rank and principal rank. Since $\text{prk}(\Gamma) = \text{order-type}(I)$ is related to the set of components of Γ , it gives a more precise idea about the structure of Γ as an ordered group than $\text{rk}(\Gamma) = \text{order-type}(\text{Cvx})$.

However, the ordered sets I and Cvx determine one to each other.

Lemma 3.6. *The set I is the subset of $\text{Cvx}\infty$ formed by all elements admitting an immediate predecessor.¹*

¹The ordering of I is the opposite of the ordering induced by $\text{Cvx}\infty$.

Proof. We have already mentioned that any non-zero principal convex subgroup H has an immediate predecessor H^* .

Conversely, if $H' \subsetneq H$ is an immediate predecessor of a convex subgroup H , then H is the principal subgroup generated by any $\gamma \in H \setminus H'$. \square

To any initial segment $S \in \text{Init}(I)$, we may associate the convex subgroup

$$H_S = \bigcup_{i \in I, i > S} H_i.$$

Conversely, to any convex subgroup $H \subset \Gamma$, we may associate the initial segment

$$S_H = \{i \in I \mid H_i \supsetneq H\} \subset I.$$

Lemma 3.7. *The assignment $S \mapsto H_S$ determines an isomorphism of ordered sets:*

$$\text{Init}(I)^{\text{opp}} \longrightarrow \text{Cvx } \infty.$$

Its inverse is the assignment $H \mapsto S_H$.

Proof. Let us first check that the first mapping is an embedding of ordered sets:

$$T \supsetneq S \implies H_T \subsetneq H_S.$$

In fact, if $i \in T \setminus S$, then $H_T \subset H_i^* \subsetneq H_i \subset H_S$.

Finally, it is an onto map because $H = H_{S_H}$ by Lemma 3.3. \square

Let us discuss in more detail how Cvx is constructed from I .

Any $i \in I$ determines two ‘‘trivial’’ initial segments:

$$I_{\leq i} = \{j \in I \mid j \leq i\} \supsetneq I_{< i} = \{j \in I \mid j < i\}.$$

There are two more trivial initial segments, the whole set I and the empty subset \emptyset .

Finally, there are non-trivial initial segments $S \subset I$ such that neither S has a maximal element, nor $I \setminus S$ has a minimal element.

To each of these initial segments the above correspondence assigns the following convex subgroups.

initial segment	convex subgroup
I	$\{0\}$
$I_{\leq i}$	H_i^*
$I_{< i}$	H_i
S non-trivial	H_S non-principal
\emptyset	Γ

The subgroup H_i^* is principal if and only if i has an immediate successor (say $i+1$) in I . In this case, $I_{\leq i} = I_{< i+1}$ and $H_i^* = H_{i+1}$.

Thus, non-principal convex subgroups arise in a two-fold way. Either from non-trivial initial segments, or from elements $i \in I$ with no immediate successor in I .

For instance, suppose that $I = \mathbb{Q}$. Then, every non-trivial initial segment of I determines a non-principal convex subgroup parameterized by a real number. On the

other hand, every rational number q determines two convex subgroups $H_q^* \subsetneq H_q$, from which only H_q is principal.

Thus, $\text{Cvx} = \{0\} + (\mathbb{R}^{\text{opp}})^\circ$, where $(\mathbb{R}^{\text{opp}})^\circ$ is a real line with the opposite order, in which every rational number has been doubled by adding an immediate predecessor of it.

Corollary 3.8. *The following conditions are equivalent.*

- (1) Γ is a principal convex subgroup.
- (1) I has a minimal element.
- (2) Cvx has a maximal element (immediate predecessor of Γ).

Example. Since \mathbb{N} is well-ordered, Corollary 3.8 shows that the Hahn product

$$\mathbb{R}_{\text{lex}}^{\mathbb{N}} = \mathbb{R}^{\mathbb{N}}$$

is a principal convex subgroup (of itself). Its immediate predecessor is the subgroup of all elements $(a_i)_{i \in \mathbb{N}}$ with $a_1 = 0$.

On the other hand, only the finite subsets of \mathbb{N}^{opp} are well-ordered. Thus, the Hahn product

$$\mathbb{R}^{(\mathbb{N}^{\text{opp}})} = \mathbb{R}_{\text{lex}}^{\mathbb{N}^{\text{opp}}} \subsetneq \mathbb{R}^{\mathbb{N}^{\text{opp}}}$$

is not a principal convex subgroup of itself, by Corollary 3.8.

4. ARQUIMEDEAN CLASSES AND HAHN'S THEOREM

Let Γ be an ordered group. Two non-zero elements $\beta, \gamma \in \Gamma$ are *archimedeanly equivalent* if there exist $n, m \in \mathbb{N}$ such that

$$|\beta| < n|\gamma| \quad \text{and} \quad |\gamma| < m|\beta|.$$

In this case, we write $\beta \sim \gamma$.

Clearly, this defines an equivalence relation on $\Gamma \setminus \{0\}$. The equivalence classes are in canonical bijection with the set $I(\Gamma)$ parameterizing non-zero principal convex subgroups of Γ .

Lemma 4.1. *Two non-zero elements $\beta, \gamma \in \Gamma$ are archimedeanly equivalent if and only if they generate the same convex subgroup: $H(\beta) = H(\gamma)$.*

Definition. We say that Γ is *archimedean* if all non-zero elements are archimedeanly equivalent.

Theorem 4.2. *Let Γ be a non-trivial ordered group. The following conditions are equivalent.*

- (1) Γ is archimedean.
- (2) Γ has rank one.
- (3) Γ is isomorphic to a subgroup of \mathbb{R} .

Proof. By Lemma 4.1, (1) is equivalent to $\text{prk}(\Gamma) = 1$. By Lemma 3.7, this is equivalent to $\text{rk}(\Gamma) = 1$ too.

By Lemma 3.5, (3) implies (2). Thus, it remains only to show that (1) implies (3).

Suppose that Γ is archimedean, and choose any positive $\gamma \in \Gamma$. The embedding $\Gamma \hookrightarrow \mathbb{R}$ sends any $\beta \in \Gamma$ to the real number determined by the sequence of rational numbers n/m such that $m\beta \leq n\gamma$. \square

Maximal groups. An ordered group Γ is *maximal* if it admits no immediate extensions. More precisely, if any immediate extension of Γ is an isomorphism.

Theorem 4.3. *Let I be an ordered set and $(C_i)_{i \in I}$ a family of ordered groups of rank one, parameterized by I . The Hahn product $(\prod_{i \in I} C_i)_{\text{lex}}$ is maximal.*

Hahn's theorem.

Definition. An ordered group Γ is *regular* if for every $i \in I(\Gamma)$, there exists a ring $\mathbb{Z} \subset A_i \subset \mathbb{Q}$ such that the component $C_i(\Gamma)$ is free as an A_i -module.

Theorem 4.4 (Hahn's theorem). *Every regular ordered group Γ admits an immediate embedding to the Hahn product determined by the skeleton of Γ .*

Corollary 4.5. *Every regular and maximal ordered group is isomorphic to the Hahn product determined by its skeleton.*

Corollary 4.6. *Every ordered group Γ may be embedded into $\mathbb{R}_{\text{lex}}^{I(\Gamma)}$.*

Proof. We may embed Γ into $\Gamma_{\mathbb{Q}}$, which is obviously regular, because its components are \mathbb{Q} -vector spaces. This extension preserves the ordered sets of principal convex subgroups: $I(\Gamma) \simeq I(\Gamma_{\mathbb{Q}})$.

By Hahn's theorem, $\Gamma_{\mathbb{Q}}$ may be embedded into a Hahn product where the components are parameterized by $I(\Gamma)$. This Hahn product may be obviously embedded into $\mathbb{R}_{\text{lex}}^{I(\Gamma)}$. \square

Lecture II. Small extensions of ordered groups

5. SMALL EXTENSIONS IN A FIXED UNIVERSE

5.1. Small extensions.

Commensurable extensions. The *rational rank* of an abelian group G is the cardinality of any maximal subset of \mathbb{Z} -linearly independent elements in G :

$$\text{rr}(G) = \dim_{\mathbb{Q}}(G \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Clearly, $\text{rr}(G) = 0$ if and only if G is a torsion group.

An extension of ordered groups $\Gamma \hookrightarrow \Lambda$ is *commensurable* if $\text{rr}(\Lambda/\Gamma) = 0$.

The extension $\Gamma \hookrightarrow \Gamma_{\mathbb{Q}}$ is obviously commensurable. Actually, $\Gamma_{\mathbb{Q}}$ is simultaneously the minimal divisible extension of Γ and the maximal commensurable extension of Γ .

Lemma 5.1. *For any commensurable extension $\Gamma \hookrightarrow \Lambda$, there exists a unique embedding of Λ into $\Gamma_{\mathbb{Q}}$ such that the composition $\Gamma \hookrightarrow \Lambda \hookrightarrow \Gamma_{\mathbb{Q}}$ is the canonical embedding.*

For an arbitrary extension $\iota: \Gamma \hookrightarrow \Lambda$, we denote by

$$\Gamma \hookrightarrow \Lambda^{\text{com}} \subset \Lambda$$

the maximal commensurable extension of Γ in Λ ; that is,

$$\Lambda^{\text{com}} = \{\xi \in \Lambda \mid m\xi \in \iota(\Gamma), \text{ for some } m \in \mathbb{N}\}.$$

Definition. We say that $\Gamma \hookrightarrow \Lambda$ is a *small extension* if $\Lambda/\Lambda^{\text{com}}$ is a cyclic group.

Therefore, a small extension is either commensurable (if $\Lambda^{\text{com}} = \Lambda$), or it has $\text{rr}(\Lambda/\Gamma) = 1$ and the quotient $\Lambda/\Lambda^{\text{com}}$ is isomorphic to \mathbb{Z} .

This definition is motivated by the following result.

Theorem 5.2. *Let K be a field and let $\mu: K[x] \rightarrow \Gamma_{\mu}\infty$ be a valuation on the polynomial ring $K[x]$. Let $\Gamma = \mu(K^*)$ be the value group of the restriction of μ to K . Then, $\Gamma \subset \Gamma_{\mu}$ is a small extension of ordered groups.*

Equivalent extensions. Two extensions of Γ ,

$$\Gamma \hookrightarrow \Lambda, \quad \Gamma \hookrightarrow \Lambda',$$

are said to be *equivalent* if there is an isomorphism $\Lambda \xrightarrow{\sim} \Lambda'$ of ordered groups fitting into a commutative diagram:

$$\begin{array}{ccc} \Lambda & & \\ \uparrow & \searrow & \\ \Gamma & \longrightarrow & \Lambda' \end{array}$$

By Lemma 5.1, every commensurable extension of Γ is equivalent to a unique subgroup of $\Gamma_{\mathbb{Q}}$.

Extensions that increase the rank at most by one. By Lemma 3.5, any extension $\Gamma \hookrightarrow \Lambda$ of ordered groups induces two embeddings of ordered sets

$$(1) \quad \text{Cvx}(\Gamma) \hookrightarrow \text{Cvx}(\Lambda), \quad \text{PrCvx}(\Gamma) \hookrightarrow \text{PrCvx}(\Lambda).$$

The following well-known inequality is an easy consequence of Hahn's theorem:

$$(2) \quad \text{rr}(\Lambda/\Gamma) \geq \# \text{PrCvx}(\Lambda) \setminus \text{PrCvx}(\Gamma),$$

where we identify $\text{PrCvx}(\Gamma)$ with its image in $\text{PrCvx}(\Lambda)$ under the embedding of (1).

Lemmas 3.6 and 3.7 describe how the sets $\text{Cvx}(\Gamma)$ and $\text{PrCvx}(\Gamma)$ determine one to each other. From this relationship it follows

$$\# \text{PrCvx}(\Lambda) \setminus \text{PrCvx}(\Gamma) = 0 \iff \# \text{Cvx}(\Lambda) \setminus \text{Cvx}(\Gamma) = 0$$

$$\# \text{PrCvx}(\Lambda) \setminus \text{PrCvx}(\Gamma) = 1 \iff \# \text{Cvx}(\Lambda) \setminus \text{Cvx}(\Gamma) = 1$$

Definition. We say that the extension $\Gamma \hookrightarrow \Lambda$ *increases the rank at most by one* if

$$\# \text{PrCvx}(\Lambda) \setminus \text{PrCvx}(\Gamma) \leq 1.$$

If $\# \text{PrCvx}(\Lambda) \setminus \text{PrCvx}(\Gamma) = 0$ we say that $\Gamma \hookrightarrow \Lambda$ *preserves the rank*.

If $\# \text{PrCvx}(\Lambda) \setminus \text{PrCvx}(\Gamma) = 1$ we say that $\Gamma \hookrightarrow \Lambda$ *increases the rank by one*.

Caution! This terminology abuses of language. If $\Gamma \hookrightarrow \Lambda$ preserves the rank, then obviously $\text{rk}(\Gamma) = \text{rk}(\Lambda)$, but the converse is not true.

For instance, $\mathbb{N}_0 = \{0\} + \mathbb{N}$ is isomorphic to \mathbb{N} as an ordered set; hence, the ordered groups $\mathbb{R}_{\text{lex}}^{\mathbb{N}}$ and $\mathbb{R}_{\text{lex}}^{\mathbb{N}_0}$ have the same rank. However, the natural embedding $\mathbb{R}_{\text{lex}}^{\mathbb{N}} \hookrightarrow \mathbb{R}_{\text{lex}}^{\mathbb{N}_0}$ increases the rank by one.

It follows from (2) that the extension $\Gamma \hookrightarrow \Gamma_{\mathbb{Q}}$ preserves the rank.

Lemma 5.3. *Every small extension $\Gamma \hookrightarrow \Lambda$ increases the rank at most by one*

Proof. Since a small extension satisfies $\text{rr}(\Lambda/\Gamma) \leq 1$, the statement is a consequence of the inequality in (2). \square

5.2. Small subextensions of a fixed universe. From now on, we fix an extension $\Gamma \hookrightarrow U$ of ordered groups, and we identify Γ with its image in U .

For any $\gamma \in U$, the subgroups generated by γ over Γ and U^{com} ,

$$\Gamma \subset \langle \Gamma, \gamma \rangle \subset \langle U^{\text{com}}, \gamma \rangle \subset U$$

are small extensions of Γ in U .

We are not aiming at a classification of the small extensions of Γ in U . Rather, in view of the applications to valuations, we are interested in the classification of the elements in U by a certain equivalence relation.

Definition. We say that $\beta, \gamma \in U$ are Γ -*equivalent* if there exists an isomorphism of ordered groups

$$\langle \Gamma, \beta \rangle \xrightarrow{\sim} \langle \Gamma, \gamma \rangle,$$

which acts as the identity on Γ and sends β to γ .

In this case, we write $\beta \sim_{\Gamma} \gamma$, or simply $\beta \sim \gamma$ if the base group Γ is clear from the context. We denote by $[\beta]_{\Gamma} = [\beta] \subset U$ the class of β .

By Lemma 5.1, any group homomorphism $\langle U^{\text{com}}, \beta \rangle \rightarrow U$ which acts as the identity on Γ , acts as the identity on U^{com} .

The next result follows immediately from this fact.

Lemma 5.4. (1) *Two elements $\beta, \gamma \in U$ are Γ -equivalent if and only if they are U^{com} -equivalent.*
 (2) *If $\beta \in U^{\text{com}}$, then $[\beta] = \{\beta\}$.*

Let $U^{\text{incom}} = U \setminus U^{\text{com}}$ be the subset of incommensurable elements over Γ . There is an easy criterion to decide when two elements in U^{incom} are Γ -equivalent.

Lemma 5.5. *Let $\beta, \gamma \in U^{\text{incom}}$ with $\beta < \gamma$. Then, β and γ are Γ -equivalent if and only if there is no $b \in U^{\text{com}}$ such that $\beta < b < \gamma$.*

Proof. Any element in the subgroup $\langle \Gamma, \beta \rangle$ may be written in a unique way as

$$a + m\beta, \quad \text{with } a \in \Gamma, \quad m \in \mathbb{Z}.$$

Hence, for any $\beta, \gamma \in U^{\text{incom}}$ there is a unique group isomorphism

$$h: \langle \Gamma, \beta \rangle \xrightarrow{\sim} \langle \Gamma, \gamma \rangle$$

acting as the identity on Γ and sending β to γ . We have $\beta \sim \gamma$ if and only if this homomorphism h preserves the ordering.

Suppose that $\beta < b < \gamma$, for some $b \in U^{\text{com}}$. Then, h does not preserve the ordering, because $\gamma = h(\beta) > b = h(b)$. Thus, β and γ are not Γ -equivalent.

Conversely, suppose that there is no $b \in U^{\text{com}}$ such that $\beta < b < \gamma$. Let us check that the homomorphism h preserves the ordering.

For arbitrary elements in $\langle \Gamma, \beta \rangle$, we clearly have,

$$(3) \quad a + m\beta < a' + m'\beta \iff \begin{cases} m = m', a < a', \text{ or} \\ m < m', (a - a')/(m' - m) < \beta, \text{ or} \\ m > m', (a - a')/(m' - m) > \beta. \end{cases}$$

By our assumption, if $m \neq m'$ we have

$$\frac{a - a'}{m' - m} < \beta \iff \frac{a - a'}{m' - m} < \gamma,$$

because $(a - a')/(m' - m)$ belongs to U^{com} .

Hence, the conditions of the right-hand side of (3) are satisfied if we replace β with γ . Therefore, $a + m\gamma < a' + m'\gamma$, and this proves that h preserves the ordering. \square

Our aim in this lecture is to find explicit computations of the quotient set U/\sim for some concrete ordered groups U .

6. SMALL EXTENSIONS THAT PRESERVE THE RANK

Let Γ be an ordered group. From now on, we denote

$$I = \text{PrCvx}(\Gamma) = \text{PrCvx}(\Gamma_{\mathbb{Q}}),$$

where we identify $\text{PrCvx}(\Gamma) = \text{PrCvx}(\Gamma_{\mathbb{Q}})$ by the natural isomorphism induced by the embedding $\Gamma \hookrightarrow \Gamma_{\mathbb{Q}}$.

If $(I; (C_i)_{i \in I})$ is the skeleton of Γ , then the skeleton of $\Gamma_{\mathbb{Q}}$ is

$$(I; (Q_i)_{i \in I}), \quad \text{where } Q_i = C_i \otimes_{\mathbb{Z}} \mathbb{Q} \text{ for all } i.$$

In section 4, we constructed a maximal embedding of Γ which preserves the rank. Let us recall this construction. Consider the Hahn product

$$\mathbb{H}(\Gamma_{\mathbb{Q}}) = \left(\prod_{i \in I} Q_i \right)_{\text{lex}}.$$

By Hahn's theorem, there is an immediate embedding $\Gamma_{\mathbb{Q}} \hookrightarrow \mathbb{H}(\Gamma_{\mathbb{Q}})$. This embedding has the following property.

Lemma 6.1. *For any $i \in I$ and any $q \in Q_i$, there exists an element $b_{i,q} \in \Gamma_{\mathbb{Q}}$ whose image in $\mathbb{H}(\Gamma_{\mathbb{Q}})$ is of the form:*

$$b_{i,q} = (\cdots 00q \star \star \cdots).$$

That is, $b_{i,q} = (b_j)_{j \in I}$, with $b_i = q$ and $b_j = 0$ for all $j < i$.

For each $i \in I$ we fix, once and for all, a positive element $1^i \in Q_i$. This choice determines an embedding $Q_i \hookrightarrow \mathbb{R}$ of ordered groups, which sends our fixed element 1^i to the real number 1.

For any $q \in Q_i$, we abuse of language and denote by the same symbol $q \in \mathbb{R}$ the image of q by the embedding $Q_i \hookrightarrow \mathbb{R}$.

Thus, we get extensions

$$\Gamma \hookrightarrow \Gamma_{\mathbb{Q}} \hookrightarrow \mathbb{H}(\Gamma_{\mathbb{Q}}) \hookrightarrow \mathbb{R}_{\text{lex}}^I.$$

Lemma 6.2. *For any rank-preserving extension $\Gamma \hookrightarrow \Lambda$, there exists an embedding $\Lambda \hookrightarrow \mathbb{R}_{\text{lex}}^I$ fitting into a commutative diagram*

$$\begin{array}{ccc} \Lambda & & \\ \uparrow & \searrow & \\ \Gamma & \longrightarrow & \mathbb{R}_{\text{lex}}^I \end{array}$$

Proof. Since $\Lambda_{\mathbb{Q}}$ is divisible, Lemma 2.1 shows that there exists a commutative diagram of embeddings of ordered groups

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Lambda_{\mathbb{Q}} \\ \uparrow & & \uparrow \\ \Gamma & \longrightarrow & \Gamma_{\mathbb{Q}} \end{array}$$

By hypothesis, the isomorphisms of (1) determine natural identifications:

$$\text{PrCvx}(\Gamma_{\mathbb{Q}}) = \text{PrCvx}(\Gamma) = I = \text{PrCvx}(\Lambda) = \text{PrCvx}(\Lambda_{\mathbb{Q}}).$$

Also, if the skeleton of $\Lambda_{\mathbb{Q}}$ is $(I; (L_i)_{i \in I})$, we have natural embeddings of \mathbb{Q} -vector spaces

$$Q_i \hookrightarrow L_i, \quad \text{for all } i \in I.$$

The immediate embedding $\Gamma_{\mathbb{Q}} \hookrightarrow \mathbb{H}(\Gamma_{\mathbb{Q}})$ given by Hahn's theorem relies on certain choices of \mathbb{Q} -bases of the components Q_i . Analogous choices for the components D_i

can be made in a compatible way with the embeddings $Q_i \hookrightarrow L_i$. Hence, we may extend our commutative diagram above to a larger one:

$$\begin{array}{ccccc} \Lambda & \longrightarrow & \Lambda_{\mathbb{Q}} & \longrightarrow & \mathbb{H}(\Lambda_{\mathbb{Q}}) \\ \uparrow & & \uparrow & & \uparrow \\ \Gamma & \longrightarrow & \Gamma_{\mathbb{Q}} & \longrightarrow & \mathbb{H}(\Gamma_{\mathbb{Q}}) \end{array}$$

Finally, for any $i \in I$, we denote by $1^i \in L_i$ the image of the positive element $1^i \in Q_i$ by the embedding $Q_i \hookrightarrow L_i$. This choice determines an embedding $L_i \hookrightarrow \mathbb{R}$ which is compatible with the embeddings $Q_i \hookrightarrow L_i$ and $Q_i \hookrightarrow \mathbb{R}$. In other words, we may assume that the composition

$$Q_i \hookrightarrow L_i \hookrightarrow \mathbb{R}$$

is our fixed embedding $Q_i \hookrightarrow \mathbb{R}$. Therefore, we have a commutative diagram of embeddings

$$\begin{array}{ccccccc} \Lambda & \longrightarrow & \Lambda_{\mathbb{Q}} & \longrightarrow & \mathbb{H}(\Lambda_{\mathbb{Q}}) & & \\ \uparrow & & \uparrow & & \uparrow & \searrow & \\ \Gamma & \longrightarrow & \Gamma_{\mathbb{Q}} & \longrightarrow & \mathbb{H}(\Gamma_{\mathbb{Q}}) & \longrightarrow & \mathbb{R}_{\text{lex}}^I \end{array}$$

This ends the proof of the lemma. \square

Caution! The embedding $\Lambda \hookrightarrow \mathbb{R}_{\text{lex}}^I$ is not necessarily unique. Thus, every rank-preserving extension of Γ is equivalent to some subextension of $\Gamma \hookrightarrow \mathbb{R}_{\text{lex}}^I$, but not to a unique one!

For instance, if $\beta, \gamma \in \mathbb{R}_{\text{lex}}^I$ are two different incommensurable elements (over Γ) which are Γ -equivalent, then the subgroups $\langle \Gamma, \beta \rangle$ and $\langle \Gamma, \gamma \rangle$ are equivalent, but they may be different.

Our aim in this section is to find a canonical system of representatives of $\mathbb{R}_{\text{lex}}^I / \sim$. Clearly,

$$(\mathbb{R}_{\text{lex}}^I)^{\text{com}} = \Gamma_{\mathbb{Q}},$$

and the classes of commensurable elements are computed in Lemma 5.4.

Thus, we focus on the computation of classes of incommensurable elements.

Canonical system of representatives of $(\mathbb{R}_{\text{lex}}^I)^{\text{incom}} / \sim$. Let $\text{Init}(I)$ be the set of initial segments of I . For any $S \in \text{Init}(I)$, consider the canonical projection

$$\pi_S: \mathbb{R}_{\text{lex}}^I \longrightarrow \mathbb{R}_{\text{lex}}^S, \quad \beta = (\beta_i)_{i \in I} \longmapsto \beta_S = (\beta_i)_{i \in S}.$$

This is a homomorphism of ordered groups, admitting a section

$$\iota_S: \mathbb{R}_{\text{lex}}^S \hookrightarrow \mathbb{R}_{\text{lex}}^I, \quad \rho = (\rho_i)_{i \in S} \longmapsto \iota_S(\rho) = (\rho \mid 0),$$

where $(\rho \mid 0)$ has the obvious meaning.

Definition. An element $\rho \in \mathbb{R}_{\text{lex}}^S$ is said to be *commensurable* over Γ if there exists $b \in \Gamma_{\mathbb{Q}}$ such that $b_S = \rho$.

Lemma 6.3. *For any $\beta \in (\mathbb{R}_{\text{lex}}^I)^{\text{incom}}$ and any $S \in \text{Init}(I)$ such that β_S is incommensurable, we have $\beta \sim (\beta_S \mid 0)$.*

Proof. Suppose that $\beta < (\beta_S \mid 0)$. Any $\gamma \in \mathbb{R}_{\text{lex}}^I$ such that $\beta < \gamma < (\beta_S \mid 0)$ has necessarily $\gamma_S = \beta_S$. Since this element is incommensurable, γ cannot belong to $\Gamma_{\mathbb{Q}}$. By the criterion of Lemma 5.5, $\beta \sim (\beta_S \mid 0)$.

If $\beta > (\beta_S \mid 0)$, the argument is completely analogous. \square

For the construction of a canonical system of representatives of $(\mathbb{R}_{\text{lex}}^I)^{\text{incom}} / \sim$ it suffices to consider inside each class $[\beta]$ the element having minimal support.

Definition. A *minimal incommensurable* element is any $\beta \in (\mathbb{R}_{\text{lex}}^I)^{\text{incom}}$ for which there exists $S \in \text{Init}(I)$ such that

- (1) β_S is incommensurable.
- (2) β_T is comensurable, for all $T \in \text{Init}(I)$ such that $T \subsetneq S$.
- (3) $\beta = (\beta_S \mid 0)$.

Theorem 6.4. *The set of minimal incommensurable elements is a system of representatives of $(\mathbb{R}_{\text{lex}}^I)^{\text{incom}} / \sim$.*

The proof of this theorem follows from Lemmas 6.6 and 6.7 below.

Lemma 6.5. *For any well-ordered set J , there is an isomorphism of ordered sets:*

$$J_{\infty} \longrightarrow \text{Init}(J), \quad j \longmapsto J_{<j} = \{k \in J \mid k < j\}$$

In particular, $\text{Init}(J)$ is a well-ordered set.

Proof. Clearly, this mapping strictly preserves the ordering:

$$j < k \implies J_{<j} \subsetneq J_{<k}.$$

Let us check that the mapping is onto. Since J is well-ordered, for any $S \in \text{Init}(J)$, $S \subsetneq J$, there exists $j_0 = \text{Min}(J \setminus S)$, and clearly $S = J_{<j_0}$.

Finally, for $S = J$ we have obviously $J = J_{<\infty}$. \square

Lemma 6.6. *For any $\beta \in (\mathbb{R}_{\text{lex}}^I)^{\text{incom}}$, the subset of $\text{Init}(I)$ formed by the initial segments S such that β_S is incommensurable contains a minimal element.*

For this minimal S , $(\beta_S \mid 0)$ is a minimal incommensurable element in $[\beta]$.

Proof. Let $J = \text{supp}(\beta)$, which is a well-ordered subset of I . By Lemma 6.5, $\text{Init}(J)$ is a well-ordered set too. Consider the natural embedding

$$\text{Init}(J) \hookrightarrow \text{Init}(I), \quad T \longmapsto \tilde{T} = \bigcup_{j \in J} I_{\leq j},$$

where \tilde{T} is the minimal initial segment of I containing T .

Consider the subsets

$$\Sigma_J = \{T \in \text{Init}(J) \mid \beta_{\tilde{T}} \text{ is incommensurable}\} \subset \text{Init}(J),$$

$$\Sigma_I = \{S \in \text{Init}(I) \mid \beta_S \text{ is incommensurable}\} \subset \text{Init}(I).$$

By definition, if $T \in \Sigma_J$, then $\tilde{T} \in \Sigma_I$.

Since β is incommensurable, J belongs to Σ_J , so that $\Sigma_J \neq \emptyset$. Since $\text{Init}(J)$ is well-ordered, there exists $T_0 = \text{Min}(\Sigma_J)$. Let us show that $\tilde{T}_0 = \text{Min}(\Sigma_I)$.

For any $S \in \Sigma_I$, the set $T = S \cap J \in \text{Init}(J)$ satisfies $\tilde{T} \subset S$. Since $\beta_{\tilde{T}}$ and β_S have support T , we have $T \in \Sigma_J$ too. Since $T_0 \subset T$, we deduce that $\tilde{T}_0 \subset \tilde{T} \subset S$.

Clearly, $(\beta_S \mid 0)$ is a minimal incommensurable element, and $\beta \sim (\beta_S \mid 0)$ by Lemma 6.3. \square

Lemma 6.7. *The minimal incommensurable elements are pairwise inequivalent.*

Proof. Let $\beta = (\beta_S \mid 0)$, $\gamma = (\gamma_T \mid 0)$ be two minimal incommensurable elements, and take $j = \text{Min}(\text{supp}(\beta - \gamma))$. Assume for instance $S \subset T$.

If $j \notin S$, then $j > S$ and $\beta_S = \gamma_S$. This implies $S = T$ and $\beta = \gamma$. In fact, since $\gamma_S = \beta_S$ is incommensurable, we cannot have $S \subsetneq T$ by the minimality of T .

Suppose $j \in S$ and let $R = I_{<j} \in \text{Init}(I)$. Since $R \subsetneq S$, $\gamma_R = \beta_R$ is commensurable. Let $b = (b_i)_{i \in I} \in \Gamma_{\mathbb{Q}}$ such that $\gamma_R = \beta_R = b_R$.

On the other hand, we have (for instance) $\beta_j < \gamma_j$, and there exists $q \in Q_j$ such that $\beta_j < q < \gamma_j$. Now, consider the element $b_{j,q-b_j} \in \Gamma_{\mathbb{Q}}$ defined in Lemma 6.1. The element $c = b + b_{j,q-b_j} \in \Gamma_{\mathbb{Q}}$ satisfies $\beta < c < \gamma$. By Lemma 5.5, β and γ are not Γ -equivalent. \square

This ends the proof of Theorem 6.4.

Definition. Let $\text{EqRk}(\Gamma)$ be the set of minimal incommensurable elements in $\mathbb{R}_{\text{lex}}^I$.

We define the *equal-rank closure* of Γ as the totally ordered set

$$\Gamma_{\mathbb{R}} = \Gamma_{\mathbb{Q}} \sqcup \text{EqRk}(\Gamma) \subset \mathbb{R}_{\text{lex}}^I,$$

which is a canonical system of representatives of $\mathbb{R}_{\text{lex}}^I / \sim$.

The set $\text{EqRk}(\Gamma)$, canonical system of representatives of $(\mathbb{R}_{\text{lex}}^I)^{\text{incom}} / \sim$, splits in a natural way into the disjoint union of two subsets.

Lemma 6.8. *Let $\beta = (\beta_S \mid 0) \in \text{EqRk}(\Gamma)$ be a minimal incommensurable element in $\mathbb{R}_{\text{lex}}^I$. The following conditions are equivalent.*

- (1) β does not belong to $\mathbb{H}(\Gamma_{\mathbb{Q}})$.
- (2) The initial segment S contains a maximal element.

Proof. Suppose that $\beta = (\beta_i)_{i \in I} \notin \mathbb{H}(\Gamma_{\mathbb{Q}})$, and let $J = \text{supp}(\beta)$. By our assumption,

$$J^0 := \{j \in J \mid \beta_j \notin Q_j\} \neq \emptyset.$$

Since J is well-ordered, it exists $j_0 = \text{Min}(J^0)$. Since $\beta_{j_0} \neq 0$, we have $j_0 \in S$. Let

$$R_0 = I_{<j_0} \subsetneq R = I_{\leq j_0} \subset S.$$

Since β_R is incommensurable and β_{R_0} is commensurable, we must have $R = S$ by the minimality of S . Hence, S has j_0 as its maximal element.

Suppose that S contains a maximal element i , and let $T = I_{<i} \subsetneq S$. By hypothesis, β_T is commensurable, so that there exists $b \in \Gamma_{\mathbb{Q}}$ such that $\beta_T = b_T$. In particular, $\beta_j \in Q_j$ for all $j < i$. This implies that $\beta_i \notin Q_i$, so that β does not belong to $\mathbb{H}(\Gamma_{\mathbb{Q}})$.

In fact, if $\beta_i \in Q_i$, we may consider the element $c = b_{i,\beta_i-b_i} \in \Gamma_{\mathbb{Q}}$ described in Lemma 6.1. The element $b+c \in \Gamma_{\mathbb{Q}}$ satisfies $(b+c)_S = \beta_S$, which is a contradiction. \square

Definition. Let $\beta \in (\mathbb{R}_{\text{lex}}^I)^{\text{incom}}$. The class $[\beta]$ is said to be *rationally incommensurable* if there exists $\gamma \in \mathbb{H}(\Gamma_{\mathbb{Q}})$ such that $\beta \sim \gamma$.

Otherwise, the class $[\beta]$ is said to be *irrationally incommensurable*.

According to this definition, we may split the set $\text{EqRk}(\Gamma)$ into the disjoint union of two subsets

$$\text{EqRk}(\Gamma) = \text{EqRk}^{\text{rat}}(\Gamma) \sqcup \text{EqRk}^{\text{irrat}}(\Gamma),$$

which represent the rationally and irrationally incommensurable classes, respectively.

By Lemma 6.8, we may describe these subsets as:

$$\text{EqRk}^{\text{rat}}(\Gamma) = \text{EqRk}(\Gamma) \cap \mathbb{H}(\Gamma_{\mathbb{Q}}), \quad \text{EqRk}^{\text{irrat}}(\Gamma) = \bigcup_{i \in I} \text{EqRk}^{\text{irrat}}(\Gamma)_i,$$

$$\text{EqRk}^{\text{irrat}}(\Gamma)_i := \left\{ (b \mid q \mid 0) \mid b \in \mathbb{R}_{\text{lex}}^{I < i} \text{ commensurable, } q \in \mathbb{R} \setminus Q_i \right\}.$$

Caution! The set $\text{EqRk}^{\text{rat}}(\Gamma)$ is a system of representatives of $(\mathbb{H}(\Gamma_{\mathbb{Q}}) \setminus \Gamma_{\mathbb{Q}}) / \sim$.

However, the set $\text{EqRk}^{\text{irrat}}(\Gamma)$ does not represent all classes in $(\mathbb{R}_{\text{lex}}^I \setminus \mathbb{H}(\Gamma_{\mathbb{Q}})) / \sim$. In this latter set we may have rationally incommensurable classes. That is, there may exist elements $\beta \in \mathbb{R}_{\text{lex}}^I \setminus \mathbb{H}(\Gamma_{\mathbb{Q}})$ such that the class $[\beta]$ contains elements in $\mathbb{H}(\Gamma_{\mathbb{Q}})$.

Before giving some examples, let us emphasize a relevant observation, which is an immediate consequence of the fact that $\Gamma_{\mathbb{Q}} = \mathbb{H}(\Gamma_{\mathbb{Q}})$ if Γ has finite rank.

Lemma 6.9. *If $\text{rk}(\Gamma) < \infty$, then $\text{EqRk}^{\text{rat}}(\Gamma) = \emptyset$.*

Examples.

(0) $\Gamma = \{0\}$.

$$I = \emptyset, \quad \Gamma_{\mathbb{Q}} = \Gamma_{\mathbb{R}} = \{0\}.$$

(1) $\text{rk}(\Gamma) = 1$.

$$\text{EqRk}^{\text{rat}} = \emptyset, \quad \text{EqRk}^{\text{irrat}} = \mathbb{R} \setminus \Gamma_{\mathbb{Q}}, \quad \Gamma_{\mathbb{R}} = \mathbb{R}.$$

(2) $\Gamma = \mathbb{R}_{\text{lex}}^2$.

$$\text{EqRk}^{\text{rat}} = \text{EqRk}^{\text{irrat}} = \emptyset, \quad \Gamma_{\mathbb{R}} = \Gamma.$$

(3) $\Gamma = \mathbb{Q}_{\text{lex}}^2$.

$$\text{EqRk}^{\text{rat}} = \emptyset, \quad \text{EqRk}^{\text{irrat}} = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R} \setminus \mathbb{Q}\} \sqcup (\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})).$$

$$\Gamma_{\mathbb{R}} = \mathbb{R}_{\text{lex}}^2 \setminus \{(x, y) \in \mathbb{R}^2 \mid x \notin \mathbb{Q}, y \neq 0\}.$$

(4) $\Gamma = \mathbb{R}^{(\mathbb{N})}$.

$$\text{EqRk}^{\text{rat}} = \mathbb{R}^{\mathbb{N}} \setminus \mathbb{R}^{(\mathbb{N})}, \quad \text{EqRk}^{\text{irrat}} = \emptyset, \quad \Gamma_{\mathbb{R}} = \mathbb{R}_{\text{lex}}^{\mathbb{N}} = \mathbb{R}^{\mathbb{N}}.$$

(5) $\Gamma = \mathbb{Q}^{(\mathbb{N})}$, $\mathbb{R}_{\text{lex}}^I = \mathbb{R}^{\mathbb{N}}$.

$$\text{EqRk}^{\text{rat}} = \mathbb{Q}^{\mathbb{N}} \setminus \mathbb{Q}^{(\mathbb{N})}, \quad \text{EqRk}^{\text{irrat}} = \bigcup_{i \in \mathbb{N}} (\mathbb{Q}^{i-1} \times (\mathbb{R} \setminus \mathbb{Q}) \times \{0\}^{N > i}).$$

$$\Gamma_{\mathbb{R}} = \mathbb{Q}^{\mathbb{N}} \sqcup \text{EqRk}^{\text{irrat}}.$$

(6) $\Gamma = \mathbb{R}^{(I)}$, for $I = \mathbb{N} + \mathbb{N} \simeq (\{1, 2\} \times \mathbb{N})_{\text{lex}}$.

$$\mathbb{R}_{\text{lex}}^I = (\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}})_{\text{lex}}, \quad \text{EqRk}^{\text{irrat}} = \emptyset.$$

$$\text{EqRk}^{\text{rat}} = \{(x, 0) \mid x \in \mathbb{R}^{\mathbb{N}} \setminus \mathbb{R}^{(\mathbb{N})}\} \sqcup \{(x, y) \mid x \in \mathbb{R}^{(\mathbb{N})}, y \in \mathbb{R}^{\mathbb{N}} \setminus \mathbb{R}^{(\mathbb{N})}\}.$$

$$\Gamma_{\mathbb{R}} = (\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}})_{\text{lex}} \setminus \{(x, y) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \mid x \notin \mathbb{R}^{(\mathbb{N})}, y \neq 0\}.$$

7. SMALL EXTENSIONS THAT INCREASE THE RANK BY ONE

7.1. One-added-element embeddings of ordered sets.

Definition. An embedding of (totally) ordered sets

$$\iota: I \hookrightarrow J$$

is a *one-added-element* embedding if $J \setminus \iota(I)$ is a one-element subset of J .

Caution! This concept should be not confused with the property

$$\text{order-type}(J) = \text{order-type}(I) + 1,$$

which has an specific meaning if I and J are well-ordered and their order-type is an ordinal number.

For instance, the one-added-element embedding

$$(4) \quad \mathbb{N} \hookrightarrow \mathbb{N}, \quad n \mapsto n + 1,$$

has source and target with order-type equal to ω .

Generic example. Let I be an ordered set. For any $S \in \text{Init}(I)$, consider the ordered set

$$(5) \quad I_S = S + \{i_S\} + S^c,$$

where $S^c = I \setminus S$ is the complementary subset of S in I .

The natural embedding $I \hookrightarrow I_S$ is a one-added-element embedding. Also, every one-added-element embedding $I \hookrightarrow J$ is isomorphic to I_S for a unique $S \in \text{Init}(I)$.

More precisely, there is a unique $S \in \text{Init}(I)$ and a unique isomorphism $I_S \xrightarrow{\sim} J$ of ordered sets, fitting into a commutative diagram

$$\begin{array}{ccc} & I_S & \\ & \uparrow \searrow & \\ I & \hookrightarrow & J \end{array}$$

For instance, for $I = J = \mathbb{N}$ and the one-added-embedding of (4), we have $S = \emptyset$, $I_S = \{0\} + \mathbb{N}$ and the isomorphism $I_S \xrightarrow{\sim} J$ maps $n \mapsto n + 1$ for all $n \geq 0$.

Universal construction. Let I be an ordered set. Consider the “double-I” set

$$II := I \cup \{i_S \mid S \in \text{Init}(I)\}.$$

We may consider a natural total ordering determined by

- (1) For all $S \in \text{Init}(I)$, the restriction of the ordering to $I_S = I \cup \{i_S\}$ is the ordering considered in (5).
- (2) $i_S < i_T \iff S \subsetneq T$, for all $S, T \in \text{Init}(I)$.

This ordered set is called the *one-added-element hull* of I . It satisfies an obvious universal property.

Lemma 7.1. *For any one-added-element embedding $I \hookrightarrow J$ of ordered sets, there exists a unique embedding $J \hookrightarrow II$ fitting into a commutative diagram*

$$\begin{array}{ccc} & J & \\ \uparrow & \searrow & \\ I & \hookrightarrow & II \end{array}$$

The image of J in II is I_S for a unique $S \in \text{Init}(I)$.

7.2. Small extensions that increase the rank by one. Let $\Gamma \hookrightarrow \Lambda$ be an extension of ordered groups.

Lemma 7.2. *If the extension $\Gamma \hookrightarrow \Lambda$ increases the rank by one, there is a unique $S \in \text{Init}(I)$ and an embedding $\Lambda \hookrightarrow \mathbb{R}_{\text{lex}}^{I_S}$ fitting into commutative diagram:*

$$\begin{array}{ccccccc} & & \Lambda & & & & \\ & \nearrow & & \searrow & & & \\ \Gamma & \hookrightarrow & \mathbb{R}_{\text{lex}}^I & \hookrightarrow & \mathbb{R}_{\text{lex}}^{I_S} & \hookrightarrow & \mathbb{R}_{\text{lex}}^{II} \end{array}$$

Proof. The initial segment S is uniquely determined by the condition $\text{PrCvx}(\Lambda) \simeq I_S$. Then, the proof follows immediately from Lemma 7.1. \square

Caution! For any $S \in \text{Init}(I)$, all subextensions of $\Gamma \hookrightarrow \mathbb{R}_{\text{lex}}^{I_S}$ increase the rank at most by one. However, the extension $\Gamma \hookrightarrow \mathbb{R}_{\text{lex}}^{II}$ admits subextensions yielding a much larger increase of the rank. We consider the ordered group $\mathbb{R}_{\text{lex}}^{II}$ only to make it clear that the union of ordered groups

$$\bigcup_{S \in \text{Init}(I)} \mathbb{R}_{\text{lex}}^{I_S} \subset \mathbb{R}_{\text{lex}}^{II}$$

has a natural total ordering.

The rest of the section is devoted to classify the incommensurable elements that increase the rank by one. The application of the criterion of Lemma 5.5, leads to the computation of a set of representatives of the quotient set

$$\left(\bigcup_{S \in \text{Init}(I)} \mathbb{R}_{\text{lex}}^{I_S} \setminus \mathbb{R}_{\text{lex}}^I \right) / \sim$$

To start with, we must determine what elements $\gamma \in \bigcup_{S \in \text{Init}(I)} (\mathbb{R}_{\text{lex}}^{I_S} \setminus \mathbb{R}_{\text{lex}}^I)$ determine an extension $\Gamma \subset \langle \Gamma, \gamma \rangle$ which increases the rank by one.

The concept of minimal incommensurable element and Lemmas 6.3, 6.6 hold in our larger groups $\mathbb{R}_{\text{lex}}^{I_S}$. However, for an element $\beta \in \mathbb{R}_{\text{lex}}^{I_S} \setminus \mathbb{R}_{\text{lex}}^I$ the class $[\beta]$ may preserve the rank if its minimal incommensurable element belongs to $\mathbb{R}_{\text{lex}}^I$.

Example. The group $\Gamma = \mathbb{Q}$ has $I = \{1\}$ and $\Gamma_{\mathbb{R}} = \mathbb{R}_{\text{lex}}^I = \mathbb{R}$. The total group $H_1 = \Gamma$ is the unique non-trivial principal convex subgroup.

For $S = I$, we have $I_S = \{1, 2\}$, where we denote $2 = i_S$ for simplicity. Thus, $\mathbb{R}_{\text{lex}}^{I_S} = \mathbb{R}_{\text{lex}}^2$, and the embedding $\mathbb{R}_{\text{lex}}^I \hookrightarrow \mathbb{R}_{\text{lex}}^2$ has image $\mathbb{R} \times \{0\}$.

The new principal subgroup of $\mathbb{R}_{\text{lex}}^2$ is $\{0\} \times \mathbb{R}$.

Let $\beta = (\sqrt{2}, \sqrt{3}) \in \mathbb{R}_{\text{lex}}^{I_S} \setminus \mathbb{R}_{\text{lex}}^I$. The subgroup generated by β is

$$\langle \Gamma, \beta \rangle = \left\{ \left(q + m\sqrt{2}, m\sqrt{3} \right) \mid q \in \mathbb{Q}, m \in \mathbb{Z} \right\}.$$

This group has rank one because $(\{0\} \times \mathbb{R}) \cap \langle \Gamma, \beta \rangle$ is the trivial group.

Alternatively, the minimal incommensurable element in the class $[\beta]$ is $\gamma = (\sqrt{2}, 0)$, which belongs to $\mathbb{R}_{\text{lex}}^I$. Thus, $\langle \Gamma, \beta \rangle \simeq \langle \Gamma, \gamma \rangle$ has rank one.

The chain of principal convex subgroups of $\mathbb{R}_{\text{lex}}^I$ is $(H_i)_{i \in I}$, where

$$H_i = \{(\beta_j)_{j \in I} \mid \beta_j = 0 \forall j < i\}.$$

Hence, if we identify Γ with its image in $\mathbb{R}_{\text{lex}}^I$, the chain of principal convex subgroups of Γ is

$$(H_i \cap \Gamma)_{i \in I}.$$

Lemma 7.3. *Let $\beta = (\beta_j)_{j \in I} \in \mathbb{R}_{\text{lex}}^{I_S} \setminus \mathbb{R}_{\text{lex}}^I$. The following conditions are equivalent.*

- (1) *The extension $\Gamma \hookrightarrow \langle \Gamma, \beta \rangle$ increases the rank by one.*
- (2) *The minimal incommensurable element in $[\beta]$ does not belong to $\mathbb{R}_{\text{lex}}^I$.*
- (3) *The minimal incommensurable element in $[\beta]$ is $(\beta_{S'} \mid 0)$, where $S' = S + \{i_S\}$.*
- (4) *The subgroup $\langle \Gamma, \beta \rangle$ contains an element $\gamma = (\gamma_i)_{i \in I_S}$ such that*

$$\gamma_{i_S} \neq 0, \quad \text{and} \quad \gamma_i = 0 \quad \text{for all } i \in S.$$

Proof. Let $T \subset I_S$ be the initial segment such that $(\beta_T \mid 0)$ is the minimal incommensurable element in $[\beta]$.

(1) \Rightarrow (2). By Lemma 6.3, the condition $(\beta_T \mid 0) \in \mathbb{R}_{\text{lex}}^I$ implies that the group $\langle \Gamma, \beta \rangle \simeq \langle \gamma, (\beta_T \mid 0) \rangle$ does not increase the rank. This contradicts (1).

(2) \Rightarrow (3). The condition $(\beta_T \mid 0) \notin \mathbb{R}_{\text{lex}}^I$ is equivalent to $\beta_{i_S} \neq 0$. In particular, $i_S \in T$ (so that $S' \subset T$) and $\beta_{S'}$ is incommensurable. By the minimality of T , we have $T = S'$.

(3) \Rightarrow (4). Since $i_S = \text{Max}(S')$, the minimality of S' implies $\beta_{i_S} \neq 0$ and β_S commensurable. Let $b \in \Gamma_{\mathbb{Q}}$ such that $\beta_S = b_S$. The element $\gamma = \beta - b$ belongs to $\langle \Gamma_{\mathbb{Q}}, \beta \rangle$ and satisfies the conditions of (4). Hence, there exists $m \in \mathbb{N}$ such that $m\gamma$ belongs to $\langle \Gamma, \beta \rangle$ and satisfies the conditions of (4).

(4) \Rightarrow (1). An element $\gamma \in \langle \Gamma, \beta \rangle$ satisfying (4) generates a new principal subgroup $H_{i_S} \cap \langle \Gamma, \beta \rangle$. The condition $\gamma_{i_S} \neq 0$ implies that γ does not belong to the smaller subgroup $\bigcup_{i > i_S} (H_i \cap \langle \Gamma, \beta \rangle)$. The condition $\gamma_i = 0$ for all $i \in S$ implies that γ does not generate any larger subgroup $H_i \cap \langle \Gamma, \beta \rangle$ for $i \in S$. \square

We may now proceed to compute a system of representatives of the subset of $(\mathbb{R}_{\text{lex}}^{I_S} \setminus \mathbb{R}_{\text{lex}}^I) / \sim$ formed by the classes that increase the rank.

By Lemma 7.3, these classes are those whose minimal incommensurable elements are

$$\beta = \beta_{S,a,q} = (a \mid q \mid 0), \quad a \in \mathbb{R}_{\text{lex}}^S \text{ commensurable}, \quad q \in \mathbb{R}^*,$$

for an arbitrary $S \in \text{Init}(I)$. Note that $q = \beta_{i_S}$.

Nevertheless, we cannot proceed as in section 6 because Lemma 6.7 fails. There are minimal incommensurable elements in $\mathbb{R}_{\text{lex}}^{I_S} \setminus \mathbb{R}_{\text{lex}}^I$ which are equivalent.

Lemma 7.4. *The minimal incommensurable elements $\beta_{S,a,q}$ and $\beta_{T,b,p}$ are Γ -equivalent if and only if*

$$(6) \quad S = T, \quad a = b, \quad pq > 0.$$

Proof. If the conditions of (6) are satisfied, we have $\beta_{S,a,q} \sim \beta_{S,a,p}$ by Lemma 5.5. In fact, for any $\gamma = (\gamma_i)_{i \in I_S} \in \mathbb{R}_{\text{lex}}^{I_S}$, the condition $\beta_{S,a,q} < \gamma < \beta_{S,a,p}$ implies $\gamma_S = a$ and $q < \gamma_{i_S} < p$. Since we are assuming that p and q have the same sign, this implies $\gamma_{i_S} \neq 0$, so that γ cannot be commensurable over Γ .

Conversely, suppose that $\beta_{S,a,q} \sim \beta_{T,b,p}$. Arguing as in the proof of Lemma 6.7, we conclude that $S = T$ and $a = b$.

Finally, suppose that p and q have a different sign; for instance, $q < 0 < p$. Then, if $\gamma = (\gamma_i)_{i \in I} \in \Gamma_{\mathbb{Q}}$ satisfies $\gamma_S = a$, we have $\gamma_{i_S} = 0$, so that $\beta_{S,a,q} < \gamma < \beta_{S,a,p}$. This is impossible by Lemma 5.5, so that p and q must have the same sign. \square

As a consequence, the classes in $\left(\bigcup_{S \in \text{Init}(I)} \mathbb{R}_{\text{lex}}^{I_S} \setminus \mathbb{R}_{\text{lex}}^I \right) / \sim$ which increase the rank are represented by the set

$$\text{IncRk}(\Gamma) = \bigcup_{S \in \text{Init}(I)} \{b^-, b^+ \mid b \in \mathbb{R}_{\text{lex}}^S \text{ commensurable}\},$$

where we define

$$b^- = \beta_{S,b,-1} = (b \mid -1 \mid 0), \quad b^+ = \beta_{S,b,1} = (b \mid 1 \mid 0).$$

If $b^{\pm} = (b_i)_{i \in I_S}$, note that $b_{i_S} = \pm 1$ and $b_j = 0$ for all $j > i_S$.

The elements corresponding to $S = \emptyset$ deserve a special notation

$$-\infty = \beta_{\emptyset,-1} = (-1 \mid 0), \quad \infty^- = \beta_{\emptyset,1} = (1 \mid 0).$$

The notation for ∞^- is motivated by the fact that this element is the immediate predecessor of ∞ in the set $\Gamma_{\text{sme}\infty}$.

Definition. The *small-extensions closure* of Γ is the ordered set

$$\Gamma_{\text{sme}} = \Gamma_{\mathbb{R}} \sqcup \text{IncRk}(\Gamma),$$

with the ordering induced by $\mathbb{R}_{\text{lex}}^{II}$.

This set is a system of representatives of $\left(\bigcup_{S \in \text{Init}(I)} \mathbb{R}_{\text{lex}}^{I_S} \right) / \sim$.

The next result follows immediately from Lemmas 6.2, 5.3 and 7.2.

Lemma 7.5. *Let $\Gamma \hookrightarrow \Lambda$ be a small extension of Γ , and choose $\gamma \in \Lambda$ such that $\Lambda = \langle \Gamma, \gamma \rangle$. For a unique $\beta \in \Gamma_{\text{sme}}$ there exists an isomorphism*

$$\langle \Gamma, \gamma \rangle \xrightarrow{\sim} \langle \Gamma, \beta \rangle$$

of ordered groups acting as the identity on Γ and sending γ to β

Therefore, in order to complete the classification of small extensions of Γ up to equivalence, we need only to classify the elements in Γ_{sme} by the equivalence relation

$$\beta \equiv \gamma \iff \langle \Gamma, \beta \rangle = \langle \Gamma, \gamma \rangle.$$

We leave this task to the reader.

Let us emphasize the position of the “increasing-rank” elements with respect to the total ordering of Γ_{sme} :

$$-\infty = \text{Min}(\Gamma_{\text{sme}}), \quad \infty^- = \text{Max}(\Gamma_{\text{sme}}).$$

For $S = I$, each $b \in \Gamma_{\mathbb{Q}}$ has an immediate predecessor and an immediate successor:

$$b^- < b < b^+, \quad b^- = \text{Max}(\Gamma_{\text{sme}, < b}), \quad b^+ = \text{Min}(\Gamma_{\text{sme}, > b}).$$

If $\emptyset \subsetneq S \subsetneq I$, then for every commensurable $b \in \mathbb{R}_{\text{lex}}^S$ we have

$$b^- < \pi_S^{-1}(b) < b^+, \quad b^- = \text{Max}(\Gamma_{\text{sme}, < \pi_S^{-1}(b)}), \quad b^+ = \text{Min}(\Gamma_{\text{sme}, > \pi_S^{-1}(b)}).$$

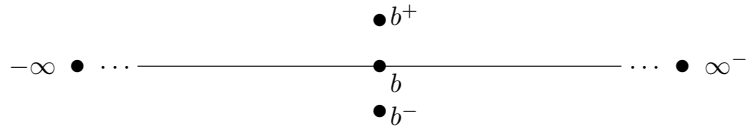
Examples. Let us draw the set Γ_{sme} in some concrete examples.

(0) $\Gamma = \{0\}$.

In this case, $\Gamma_{\mathbb{Q}} = \Gamma_{\mathbb{R}} = \{0\}$ and $\Gamma_{\text{sme}} = \{-\infty, 0, \infty^-\}$.

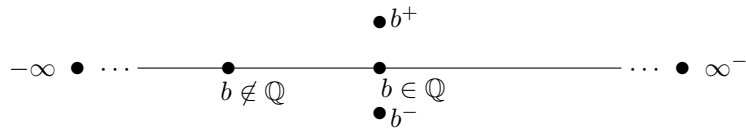
(1) $\Gamma = \mathbb{R}$.

In this case, $\Gamma_{\mathbb{Q}} = \Gamma_{\mathbb{R}} = \mathbb{R}$ and Γ_{sme} is a real line with global minimal and maximal added elements, and such that to every real number an immediate predecessor and successor have been added.



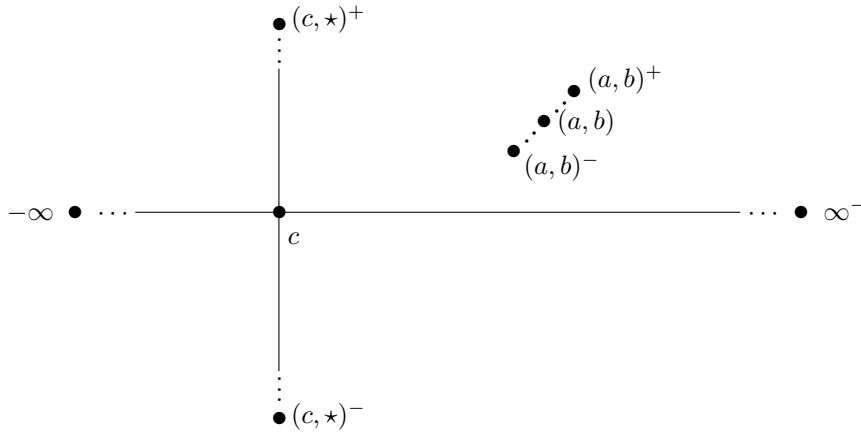
(2) $\Gamma = \mathbb{Q}$.

In this case, $\Gamma_{\mathbb{Q}} = \mathbb{Q}$, $\Gamma_{\mathbb{R}} = \mathbb{R}$ and Γ_{sme} is a real line with global minimal and maximal added elements, and such that to every rational number an immediate predecessor and successor have been added.



(3) $\Gamma = \mathbb{R}_{\text{lex}}^2$.

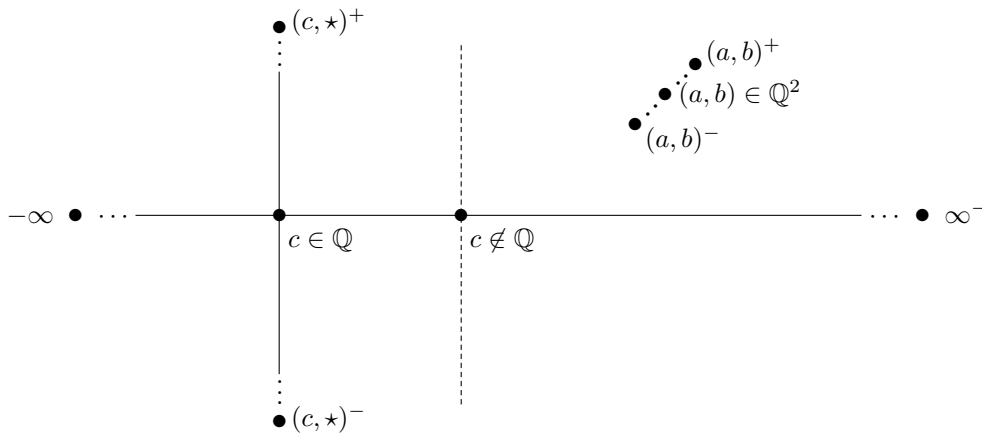
In this case, $\Gamma_{\mathbb{Q}} = \Gamma_{\mathbb{R}} = \mathbb{R}_{\text{lex}}^2$ and Γ_{sme} is a real plane with global minimal and maximal added elements, such that every vertical line has a minimal and maximal added element, and to every single point an immediate predecessor and successor have been added.



(4) $\Gamma = \mathbb{Q}_{\text{lex}}^2$.

In this case, $\Gamma_{\mathbb{Q}} = \Gamma$ and $\Gamma_{\mathbb{R}} = \mathbb{R}_{\text{lex}}^2 \setminus \{(x, y) \mid x \notin \mathbb{Q}, y \neq 0\}$.

Now, Γ_{sme} adds to $\Gamma_{\mathbb{R}}$ a global minimal and maximal elements. Also, it adds a minimal and maximal element to each vertical line with rational abscissa. Finally, it adds an immediate predecessor and successor to every single rational point in \mathbb{Q}^2 .



Lecture III. Extension of valuations to polynomial rings

8. VALUATIONS ON A POLYNOMIAL RING

Let K be a field, and let $K[x]$ be the polynomial ring in one indeterminate. A *valuation* on $K[x]$ is a mapping

$$\mu: K[x] \longrightarrow \Lambda \infty$$

where Λ is an ordered group, such that the following two conditions are satisfied for all $f, g \in K[x]$:

- (1) $\mu(fg) = \mu(f) + \mu(g)$,
- (2) $\mu(f + g) \geq \text{Min}\{\mu(f), \mu(g)\}$.

The *support* of μ is the prime ideal

$$\mathfrak{p} = \mathfrak{p}_\mu = \mu^{-1}(\infty) \in \text{Spec}(K[x]).$$

The value group of μ is the subgroup $\Gamma_\mu \subset \Lambda$ generated by $\mu(K[x] \setminus \mathfrak{p})$.

The valuation μ induces a valuation on the residue field $\kappa(\mathfrak{p})$, field of fractions of $K[x]/\mathfrak{p}$. Note that $\kappa(0) = K(x)$, while for $\mathfrak{p} \neq 0$ the field $\kappa(\mathfrak{p})$ is a simple finite extension of K .

Thus, a valuation μ on $K[x]$ determines a valuation on a simple field extension L/K , either algebraic or transcendental.

Let (K, v) be a valued field, and denote by $\Gamma = v(K^*)$ the value group.

Definition. A valuation μ on $K[x]$ is an *extension* of v if the valuation on K obtained by restriction of μ is equivalent to v . In other words, there exists an embedding of ordered groups $\iota: \Gamma \hookrightarrow \Gamma_\mu$, fitting into a commutative diagram

$$\begin{array}{ccc} K[x] & \xrightarrow{\mu} & \Gamma_\mu \infty \\ \uparrow & & \uparrow \iota \\ K & \xrightarrow{v} & \Gamma \infty \end{array}$$

In this case, the valuation induced by μ on the field $L = K[x]/\mathfrak{p}_\mu$ is an extension of v to that field.

Definition. The extension μ/v is *commensurable*, *preserves the rank*, or *increases the rank by one*, if the extension of ordered groups $\iota: \Gamma \hookrightarrow \Gamma_\mu$ has this property, respectively.

All valuations with non-trivial support are commensurable over v .

Definition. Two valuations on $K[x]$,

$$\mu: K[x] \longrightarrow \Gamma_\mu \infty, \quad \mu': K[x] \longrightarrow \Gamma_{\mu'} \infty,$$

are *equivalent* if there is an isomorphism of ordered groups $\varphi: \Gamma_\mu \xrightarrow{\sim} \Gamma_{\mu'}$ fitting into a commutative diagram

$$(7) \quad \begin{array}{ccc} \Gamma_\mu \infty & \xrightarrow{\varphi} & \Gamma_{\mu'} \infty \\ \mu \uparrow & \nearrow & \mu' \\ K[x] & & \end{array}$$

Obviously, two equivalent valuations on $K[x]$ induce equivalent valuations on K by restriction.

In particular, if μ, μ' are equivalent extensions of v , determining embeddings

$$\iota: \Gamma \hookrightarrow \Gamma_\mu, \quad \iota': \Gamma \hookrightarrow \Gamma_{\mu'},$$

the commutativity of the diagram (7) implies the commutativity of

$$\begin{array}{ccc} \Gamma_\mu & \xrightarrow{\varphi} & \Gamma_{\mu'} \\ \iota \uparrow & \nearrow & \iota' \\ \Gamma & & \end{array}$$

Usually, we shall identify Γ with its image in Γ_μ and $\Gamma_{\mu'}$. Then, the isomorphism φ will necessarily act as the identity on Γ .

Aim. Describe the set of extensions of v to $K[x]$, up to equivalence.

In this lecture we cover a few modest steps towards this aim.

Let us fix an extension $\Gamma \hookrightarrow \Lambda$ of ordered groups, and consider extensions of v ,

$$\mu: K[x] \longrightarrow \Lambda \infty,$$

which are Λ -valued. The set of these valuations admits a partial ordering:

$$\mu \leq \mu' \iff \mu(f) \leq \mu'(f) \quad \text{for all } f \in K[x].$$

This partial ordering does not behave well with respect to the equivalence of valuations. However, we have a natural partial ordering \leq on the set of equivalence classes of commensurable extensions μ/v , because these classes may be identified with $\Gamma_{\mathbb{Q}}$ -valued extensions of v , by Lemma 5.1.

9. DEPTH ZERO VALUATIONS ON $K[x]$

Definition. Let $\Gamma \hookrightarrow \Lambda$ be an embedding of ordered groups.

For given $a \in K$ and $\gamma \in \Lambda$, consider the valuation $\mu = \mu_{a,\gamma}$ on $K[x]$ defined as

$$\mu \left(\sum_{0 \leq s} a_s (x - a)^s \right) = \text{Min} \{ v(a_s) + s\gamma \mid 0 \leq s \}.$$

This extension μ of v is said to be a *depth-zero valuation* on $K[x]$.

Note that $\mu_{a,\gamma}$ has trivial support, and $\Gamma_\mu = \langle \Gamma, \gamma \rangle$ is a small extension of Γ .

We would like to classify these depth-zero extensions of v up to equivalence. In principle, the main difficulty is the fact that we have freedom in the choice of the extensions $\Gamma \hookrightarrow \Lambda$. However, for a fixed $a \in K$, the classification of depth-zero valuations is an easy consequence of the results of Lecture II.

Theorem 9.1. *Let $a \in K$. Every depth-zero valuation $\mu = \mu_{a,\gamma}$ is equivalent to $\mu_{a,\beta}$ for a unique $\beta \in \Gamma_{\text{sme}}$.*

Moreover, μ/v is commensurable if and only if $\beta \in \Gamma_{\mathbb{Q}}$. Also, μ/v preserves the rank if and only if $\beta \in \Gamma_{\mathbb{R}}$.

Proof. Regardless of the ordered groups containing the values β, γ , the depth-zero valuations $\mu_{a,\gamma}$ and $\mu_{a,\beta}$ are equivalent if and only if there exists an isomorphism of ordered groups $\varphi: \langle \Gamma, \gamma \rangle \rightarrow \langle \Gamma, \beta \rangle$ fitting into a commutative diagram

$$\begin{array}{ccc} \langle \Gamma, \gamma \rangle_{\infty} & \xrightarrow{\varphi} & \langle \Gamma, \beta \rangle_{\infty} \\ \mu_{a,\gamma} \uparrow & \nearrow \mu_{a,\beta} & \\ K[x] & & \end{array}$$

This is equivalent to φ acting as the identity on Γ and mapping γ to β .

Since $\langle \Gamma, \gamma \rangle$ is a small extension of Γ , Lemma 7.5 shows that there exists a unique $\beta \in \Gamma_{\text{sme}}$ for which these conditions hold. \square

Finally, for two values β, γ in the same ordered group Λ it is easy to check that

$$(8) \quad \mu_{a,\beta} = \mu_{b,\gamma} \iff \beta = \gamma \leq v(a - b).$$

Together with Theorem 9.1, this leads to an explicit description of the equivalence classes of depth-zero valuations.

Extreme valuations. For a fixed $a \in K$, and values $\beta, \gamma \in \Gamma_{\text{sme}}$ we clearly have

$$\mu_{a,\beta} \leq \mu_{a,\gamma} \iff \beta \leq \gamma.$$

Therefore, the set of equivalence classes $\mathcal{T}_a := \{\mu_{a,\gamma} \mid \gamma \in \Gamma_{\text{sme}}\}$ inherits an ordering from Γ_{sme} . Since Γ_{sme} has a minimal and maximal element,

$$-\infty \leq \Gamma_{\text{sme}} \leq \infty^-,$$

the corresponding valuations are minimal and maximal elements of \mathcal{T}_a , respectively.

$$\mu_{a,-\infty} \bullet \cdots \xrightarrow{\mu_{a,\gamma}} \bullet \cdots \bullet \mu_{a,\infty^-}$$

From the relationship (8) we deduce that the minimal valuation $\mu_{a,-\infty}$ is independent of a :

$$\mu_{a,-\infty} = \mu_{b,-\infty} \quad \text{for all } a, b \in K.$$

We shall denote this absolute minimal depth-zero valuation simply by $\mu_{-\infty}$.

By thinking in the case $a = 0$, we see that it acts as follows:

$$\mu_{-\infty}: K[x] \longrightarrow (\mathbb{Z} \times \Gamma)_{\infty}, \quad f \longmapsto (\text{ord}_{\infty}(f), v(\text{lc}(f))),$$

where $\text{lc}(f)$ is the leading coefficient of a non-zero polynomial f .

On the other hand, the maximal valuation μ_{a,∞^-} acts as follows:

$$\mu_{a,\infty^-}: K[x] \longrightarrow (\mathbb{Z} \times \Gamma)_{\infty}, \quad f \longmapsto (\text{ord}_{x-a}(f), v(\text{init}(f))),$$

where $\text{init}(f)$ is the first non-zero coefficient of the $(x - a)$ -expansion of a non-zero $f \in K[x]$.

10. KEY POLYNOMIALS AND AUGMENTATION OF VALUATIONS

Let μ be an extension of v to $K[x]$ with trivial support.

For arbitrary polynomials $f, g \in K[x]$, we write $f \mid_{\mu} g$ to indicate the existence of $h \in K[x]$ such that $\mu(g - fh) > \mu(g)$.

A *key polynomial* for μ is a monic polynomial $\phi \in K[x]$ satisfying two conditions, for arbitrary $f, g \in K[x]$:

- (1) $\phi \mid_{\mu} fg \implies \phi \mid_{\mu} f$ or $\phi \mid_{\mu} g$,
- (2) $\phi \mid_{\mu} f \implies \deg(\phi) \leq \deg(f)$.

A key polynomial for μ is necessarily irreducible in $K[x]$.

Let $\phi \in K[x]$ be a monic non-constant polynomial. Any $f \in K[x]$ admits a canonical ϕ -expansion

$$f = \sum_{0 \leq s} a_s \phi^s, \quad a \in K[x], \quad \deg(a) < \deg(\phi).$$

Lemma 10.1. *Let $\phi \in K[x]$ be a key polynomial for μ .*

- (1) *For all $f \in K[x]$ with ϕ -expansion $f = \sum_{0 \leq s} a_s \phi^s$ we have*

$$\mu(f) = \text{Min}\{\mu(a_s \phi^s) \mid s \geq 0\}.$$

- (2) *The following subset of Γ_{μ} is a subgroup which is commensurable over Γ .*

$$\Gamma_{\mu, \phi} := \{\mu(a) \mid a \in K[x], 0 \leq \deg(a) < \deg(\phi)\}.$$

In particular, $\Gamma_{\mu} = \langle \Gamma_{\mu, \phi}, \mu(\phi) \rangle$ is a small extension of Γ .

By eventually replacing μ with an equivalent valuation, we may suppose that $\Gamma_{\mu, \phi} \subset \Gamma_{\mathbb{Q}}$. Then, Lemma 7.5 shows the existence of a unique $\rho \in \Gamma_{\text{sme}}$ admitting an isomorphism

$$\varphi: \Gamma_{\mu} = \langle \Gamma_{\mu, \phi}, \mu(\phi) \rangle \longrightarrow \langle \Gamma_{\mu, \phi}, \rho \rangle$$

acting as the identity on $\Gamma_{\mu, \phi}$ and sending $\mu(\phi)$ to ρ .

Therefore, by eventually replacing μ with an equivalent valuation, we may suppose that $\mu(\phi) \in \Gamma_{\text{sme}}$.

Definition. Let μ be an extension of v to $K[x]$ admitting key polynomials.

Let $\Gamma_{\mu} \hookrightarrow \Lambda$ be an embedding of ordered groups.

For given $\phi \in K[x]$ a key polynomial for μ , and $\gamma \in \Lambda$ such that $\gamma > \mu(\phi)$, consider the valuation $\mu' = [\mu; \phi, \gamma]$ on $K[x]$ defined on ϕ -expansions as

$$\mu' \left(\sum_{0 \leq s} a_s \phi^s \right) = \text{Min} \{ \mu(a_s) + s\gamma \mid 0 \leq s \}.$$

This extension μ' of v is said to be an *ordinary augmentation* of μ with augmentation data ϕ, γ .

We also write $\mu \xrightarrow{\phi, \gamma} \mu'$. Note that μ' has trivial support, and $\Gamma_{\mu'} = \langle \Gamma_{\mu, \phi}, \gamma \rangle$ is a small extension of Γ .

We may construct plenty of augmented valuations of μ by considering arbitrarily large extensions Λ .

If we fix the key polynomial ϕ , these augmentations are easily classified up to equivalence.

Theorem 10.2. *Let ϕ be a key polynomial for μ . Every augmented valuation $\mu' = [\mu; \phi, \gamma]$ is equivalent to $[\mu; \phi, \beta]$ for a unique $\beta \in \Gamma_{\text{sme}}$ such that $\beta > \mu(\phi)$.*

Moreover, μ/v is commensurable if and only if $\beta \in \Gamma_{\mathbb{Q}}$. Also, μ/v preserves the rank if and only if $\beta \in \Gamma_{\mathbb{R}}$.

The proof is completely analogous to that of Theorem 9.1.

Finally, if we consider different key polynomials ϕ, χ for μ and two values $\beta, \gamma > \mu(\phi)$ in the same ordered group Λ , then we have

$$[\mu; \phi, \beta] = [\mu; \chi, \gamma] \iff \beta = \gamma \leq \mu(\phi - \chi).$$

Together with Theorem 10.2, this leads to an explicit description of the equivalence classes of valuations which may be obtained as ordinary augmentations of μ .

Extreme valuation. For a fixed key polynomial ϕ for μ , and values $\beta, \gamma \in \Gamma_{\text{sme}}$, $\beta, \gamma > \mu(\phi)$, we clearly have

$$[\mu; \phi, \beta] \leq [\mu; \phi, \gamma] \iff \beta \leq \gamma.$$

Thus, the set of equivalence classes of ordinary augmentations of μ with augmentation data ϕ :

$$\mathcal{T}_{\mu, \phi} := \{[\mu; \phi, \gamma] \mid \gamma \in \Gamma_{\text{sme}}, \gamma > \mu(\phi)\}$$

inherits an ordering from Γ_{sme} . Since Γ_{sme} has a maximal element ∞^- , this set contains a maximal element too.

$$\mu \bullet \xrightarrow{[\mu; \phi, \gamma]} \dots \bullet [\mu; \phi, \infty^-]$$

This maximal valuation $[\mu; \phi, \infty^-]$ acts as follows:

$$[\mu; \phi, \infty^-]: K[x] \longrightarrow (\mathbb{Z} \times \Gamma_{\mu}) \infty, \quad f \longmapsto (\text{ord}_{\phi}(f), \mu(\text{init}(f))),$$

where $\text{init}(f)$ is the first non-zero coefficient of the ϕ -expansion of a non-zero $f \in K[x]$.

11. LIMIT AUGMENTATIONS

Let μ be an extension of v to $K[x]$, with trivial support.

A countably infinite chain of ordinary augmentations

$$(9) \quad \mu = \rho_0 \xrightarrow{\chi_1, \beta_1} \rho_1 \xrightarrow{\chi_2, \beta_2} \dots \longrightarrow \rho_{i-1} \xrightarrow{\chi_i, \beta_i} \rho_i \longrightarrow \dots$$

is said to be a *continuous MacLane chain* of μ (abbreviated ML-chain) if it satisfies the following conditions

- (1) The key polynomials χ_i have the same degree for all $i \geq 1$.
- (2) There exists χ_0 , key polynomial for μ of minimal degree, such that $\chi_1 \nmid_{\mu} \chi_0$.
- (3) $\chi_{i+1} \nmid_{\rho_i} \chi_i$, $\forall i \geq 1$.

The constant number $m = \deg(\chi_i)$, for all $i \geq 1$, is called the *stable degree* of the continuous MacLane chain.

A polynomial $f \in K[x]$ is *stable* (with respect to the given continuous ML-chain) if there exists $i \geq 0$ such that $\rho_i(f) = \rho_{i+1}(f)$.

In this case, $\rho_i(f) = \rho_j(f)$ for all $j \geq i$. We denote by $\mu_{\infty}(f)$ this stable value of f .

We say that μ_∞ is the *stability function* of the continuous ML-chain. This function depends on the continuous ML-chain, and not only on μ .

Lemma 11.1. *Consider a continuous MacLane chain of μ with stable degree m .*

- (1) *For all $i \geq 0$, $\Gamma_{\rho_i} = \Gamma_\mu$ is commensurable over Γ .*
- (2) *All polynomials of degree less than m are stable.*
- (3) *Γ_μ coincides with the set of stable values of all stable polynomials.*
- (4) *All key polynomials χ_i are stable.*

In particular, we may suppose that $\Gamma_\mu \subset \Gamma_\mathbb{Q}$, by eventually replacing μ with an equivalent valuation.

If all polynomials in $K[x]$ are stable, then μ_∞ is a valuation on $K[x]$ with trivial support. We say that μ_∞ is the *stable limit* of the continuous ML-chain.

Definition. Suppose that all polynomials of degree m are stable, but there exist non-stable polynomials. Take a non-stable monic $\phi \in K[x]$ of minimal degree. Since the product of stable polynomials is stable, ϕ is irreducible in $K[x]$.

Let $\Gamma_\mu \hookrightarrow \Lambda$ be an embedding of ordered groups, and choose $\gamma \in \Lambda$ such that $\rho_i(\phi) < \gamma$ for all $i \geq 0$. We denote by $\mu' = [\mu_\infty; \phi, \gamma]$ the valuation

$$\mu' : K[x] \longrightarrow \Lambda \cup \{\infty\}$$

assigning to any $f \in K[x]$, with ϕ -expansion $f = \sum_{0 \leq s} a_s \phi^s$, the value

$$\mu'(f) = \text{Min}\{\mu_\infty(a_s) + s\gamma \mid 0 \leq s\}.$$

Note that μ' has trivial support. We say that μ' is a *limit-augmentation* of μ with respect to the augmentation data $(\rho_i)_{i \geq 0}$, ϕ and γ .

Lemma 11.2. (1) *If $f \in K[x]$ is a stable polynomial, then $\mu'(f) = \mu_\infty(f)$.*
 (2) *$\Gamma_{\mu'} = \langle \Gamma_\mu, \gamma \rangle$ is a small extension of Γ .*

Therefore, we may argue as in the case of ordinary augmentations.

If we fix the non-stable monic polynomial ϕ of minimal degree, the valuations that can be obtained as limit-augmentations of μ are easily classified up to equivalence.

Theorem 11.3. *Let $\phi \in K[x]$ be a non-stable monic polynomial of minimal degree $\deg(\phi) > m$, with respect to a continuous ML-chain of μ of stable degree m as in (9). Every limit-augmented valuation $\mu' = [\mu_\infty; \phi, \gamma]$ is equivalent to $[\mu_\infty; \phi, \beta]$ for a unique $\beta \in \Gamma_{\text{sme}}$ such that $\beta > \rho_i(\phi)$ for all $i \geq 0$.*

Moreover, μ/v is commensurable if and only if $\beta \in \Gamma_\mathbb{Q}$. Also, μ/v preserves the rank if and only if $\beta \in \Gamma_\mathbb{R}$.

The proof is completely analogous to that of Theorem 9.1.

Finally, if we consider different non-stable monic polynomials of minimal degree ϕ, χ , and two values β, γ in the same ordered group Λ , such that $\beta, \gamma > \rho_i(\phi)$ for all $i \geq 0$, then

$$[\mu_\infty; \phi, \beta] = [\mu_\infty; \chi, \gamma] \iff \beta = \gamma \leq \mu_\infty(\phi - \chi).$$

Together with Theorem 11.3, this leads to an explicit description of the equivalence classes of valuations which may be obtained as limit-augmentations of μ , with respect to the same ML-chain (9).

Extreme valuation. Consider a continuous ML-chain as in (9) of stable degree m , and a fixed monic non-stable ϕ of minimal degree $\deg(\phi) > m$.

For all values $\beta, \gamma \in \Gamma_{\text{sme}}$, $\beta, \gamma > \rho_i(\phi)$ for all $i \geq 0$, we clearly have

$$[\mu_\infty; \phi, \beta] \leq [\mu_\infty; \phi, \gamma] \iff \beta \leq \gamma.$$

Thus, the set of equivalence classes of limit-augmentations of μ with augmentation data $(\rho_i)_{i \geq 0}$, ϕ :

$$\mathcal{T}_{\mu, (\rho_i)_{i \geq 0}, \phi} := \{[\mu_\infty; \phi, \gamma] \mid \gamma \in \Gamma_{\text{sme}}, \gamma > \rho_i(\phi) \text{ for all } i \geq 0\}$$

inherits an ordering from Γ_{sme} . Since Γ_{sme} has a maximal element ∞^- , this set contains a maximal element too.

$$\mu \bullet \xrightarrow{[\mu_\infty; \phi, \gamma]} \bullet \cdots \bullet [\mu_\infty; \phi, \infty^-]$$

This maximal valuation $[\mu_\infty; \phi, \infty^-]$ acts as follows:

$$[\mu_\infty; \phi, \infty^-]: K[x] \longrightarrow (\mathbb{Z} \times \Gamma_\mu) \infty, \quad f \longmapsto (\text{ord}_\phi(f), \mu_\infty(\text{init}(f))),$$

where $\text{init}(f)$ is the first non-zero coefficient of the ϕ -expansion of a non-zero $f \in K[x]$.