Anderson's "cyclotomic units" and special L-values

Let p be an odd prime number and let X be the p-Sylow subgroup of the ideal class group of $\mathbb{Q}(e^{\frac{2i\pi}{p}})$. Let $\Delta = \mathrm{Gal}(\mathbb{Q}(e^{\frac{2i\pi}{p}})/\mathbb{Q}) \simeq (\frac{\mathbb{Z}}{p\mathbb{Z}})^{\times}$ and let $\widehat{\Delta} = \mathrm{Hom}(\Delta, \mathbb{Z}_p^{\times})$. Observe that X is a $\mathbb{Z}_p[\Delta]$-module. For $\chi \in \widehat{\Delta}$, let:

$$X(\chi) = e_{\chi}X,$$

where $e_{\chi} = \frac{1}{|\Delta|}\sum_{\delta\in\Delta}\chi(\delta)\delta^{-1} \in \mathbb{Z}_p[\Delta]$. Let $\omega_p \in \widehat{\Delta}$ be the p-adic Teichmüller character, and let $n \equiv 1 \pmod 2$, $n \in \{3, \ldots, p-2\}$. Then K. Ribet proved the following result ([6]):

$$X(\omega_p^n) \neq \{0\} \Leftrightarrow B_{p-n} \equiv 0 \pmod p,$$

where B_n denotes the n-th Bernoulli number.

Our aim in these lectures is to present a proof as self-contained as possible of Taelman's analogue (for the Carlitz module, see [5] chapter 3) of the above Theorem ([8]). Our exposition will follow quite closely the approach of L. Taelman and the orator in [4]; more precisely, we will show how Anderson's cyclotomic units ([1], [2]), an equivariant class formula in the spirit of [7], and some arithmetic properties of Thakur's Gauss sums ([9]) can be used to prove Taelman's Herbrand-Ribet Theorem. If we will have enough time, we will also explain how these latter ideas can be generalized for Drinfeld modules defined over Tate algebras ([3]).

References

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