Capítol 2

Parametrizations of elliptic curves by Shimura curves and by classical modular curves

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Introduction

This is an expository paper following Kenneth Ribet and Shuzo Takahashi, cf. [9]. Let $N = DM$, where D is a product of an even number of distinct primes and M is an integer prime to D. Let f be a newform in $S_2(\Gamma_0(N), \mathbb{Q})$. By Jacquet-Langlands correspondence, f corresponds to a newform f' in $S_2(\Phi_0^D(M))$, where $\Phi_0^D(M)$ is the group of norm 1 elements in an Eichler order of the quaternion algebra over $\mathbb{Q}$ of discriminant D (see for example [5]). There are elliptic curves A and A', associated to f and f' respectively, and they are covered by a modular and a Shimura curve respectively. The results in [9] compare the degrees δ and δ' of the two coverings. It is a well-known fact that these degrees have to do with congruences of f in some

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suitable spaces of modular forms. It turns out that the ratio δ/δ' can be described in terms of the orders c_p of the groups of components of the fiber at p of the Néron model of A and A', for p dividing D, and an "error term"(which the authors explicitely describe) whose support consists only of primes ℓ for which the Galois module $A[\ell]$ is reducible.

A partial generalization of this result in the case where A has non-semistable reduction at some prime ℓ has been obtained in [10, see Corollary 4.7 and the discussion below].

2.1 Degree of parametrization

Classical case

Let $f = \sum a_n q^n$ be a newform in $S_2(\Gamma_0(N), \mathbb{Q})$. Shimura associated to f an elliptic curve A over $\mathbb{Q}$, which is a quotient of $J_0(N)$:

$$\xi : J_0(N) \longrightarrow A.$$

By composing with the standard map $X_0(N) \hookrightarrow J_0(N)$ we get a covering

$$\pi : X_0(N) \to A$$

The **degree of parametrization** of A is the degree $\delta = \delta(N)$ of the covering π.

The degree δ can also be viewed in the following way: the map ξ induces on dual varieties a map

$$\check{\xi} : \check{A} \longrightarrow J_0\check{(}N)$$

jacobians of curves are canonically self dual, so that

$$\check{\xi} : A \longrightarrow J_0(N)$$

$\xi \circ \check{\xi} \in \mathrm{End}(A)$ is the multiplication by the integer δ.

Importance of δ for congruences

Primes p dividing $\delta(N)$ are **congruence primes** for f:

$$p|\delta(N) \quad \Longleftrightarrow \quad \begin{array}{l}\text{there is a Hecke eigenform } g \in S_2(\Gamma_0(N), \mathbb{Q}) \\ \text{such that } f \equiv g \mod p.\end{array}$$

(Ribet [7, 6], Zagier [11], et al. around 1980)

The quaternionic case

Suppose now $N = DM$ with $(D, M) = 1$ and D product of an even number of distinct primes, so that the quaternion algebra B over $\mathbb{Q}$ of discriminant D is undefined.

Let $R(M)$ be an Eichler order of level M in B and let $\Phi_0^D(M)$ be the group of elements of norm 1 in $R(M)$.

By Jacquet-Langlands correspondence there is a Hecke eigenform $f' \in S_2(\Phi_0^D(M))$, M-new, having the same eigenvalues as f for all the Hecke operators.

There is an abelian variety A' associated to f', isogenous to A, and a map

$$\xi' : J_0^D(M) \longrightarrow A'.$$

Then one can define the degree of this parametrization

$$\delta^D(M) = \xi' \circ \check{\xi}' \in \mathbb{Z}.$$

Interpretation of $\delta^D(M)$ in terms of congruences

$$p|\delta^D(M) \quad \Longleftrightarrow \quad \begin{array}{l}\text{there is a Hecke eigenform } g \in S_2(\Gamma_0(N), \mathbb{Q})^{D-\text{new}} \\ \text{such that } f \equiv g \mod p.\end{array}$$

Let $\Phi(A, p)$ be the group of components of the fiber at p of the Néron model of A, and

$c_p = |\Phi(A, p)| = ord_p(\Delta)$ where δ is the minimal discriminant of A.

It is known (level-lowering results, for example Ribet [8]) that c_p controls congruences between f and p-old forms in $S_2(\Gamma_0(N))$.

2.2 The main result

These considerations yield to the following heuristic formula:

$$\delta^D(M) = \frac{\delta(N)}{\prod_{p|D} c_p}$$

or (recursively), considering a factorization $N = DpqM$

$$\delta^{Dpq}(M) = \frac{\delta^D(pqM)}{c_p c_q}$$

This formula is in general FALSE.

For example consider $M = 1, D = 1, pq = 14$. There is a unique newform f in $S_2(\Gamma_0(14))$. One has $\delta(14) = 1$, $\delta^{14}(1) = 1$, $c_2 = 6$, $c_7 = 3$ (tables of Antwerp IV, [1])

To state the correct version of the formula we need some notations:

$$\begin{array}{ll} J = J_0^D(Mpq) & J' = J_0^{Dpq}(M) \\ \xi : J \to A & \xi' : J' \to A' \\ c_p = |\Phi(A,p)| & c'_p = |\Phi(A',p)| \end{array}$$

2.2.1 Teorema *1. One has*

$$\delta^{Dpq}(M) = \frac{\delta^D(pqM)}{c'_p c_q} \mathcal{E}(D,p,q,M)^2$$

where the "error term"$\mathcal{E}(D,p,q,M) \in \mathbb{Z}$ is a positive divisor of $c'_p c_q$.

2. *Suppose M is square free but not a prime number and let ℓ be a prime dividing $\mathcal{E}(D,p,q,M)$. Then the* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-module $A[\ell]$ is reducible.*

2.3 Proof of Assertion 1

In order to prove Assertion 1, the authors give an explicit description of $\mathcal{E}(D, p, q, M)$.

If V is an abelian variety over $\mathbb{Q}$ and ℓ is a prime, let

$$\Phi(V, \ell) \quad = \quad \text{group of components of the fiber at } \ell \text{ of the Néron model of } V$$

Then the following facts are known:

- $\Phi(V, \ell)$ is a finite étale group scheme over $Spec(\mathbb{F}_\ell)$, i.e. it is finite abelian with a canonical action of $\mathrm{Gal}(\overline{\mathbb{F}_\ell}/\mathbb{F}_\ell)$;
- if $V = A$ is an elliptic curve with multiplicative reduction at ℓ then $\Phi(A, \ell)$ is cyclic
- the association $V \mapsto \Phi(V, \ell)$ is functorial

The maps $\xi : J \to A$, $\xi' : J' \to A'$ induce

$$\xi_* : \Phi(J, q) \longrightarrow \Phi(A, q) \qquad \xi'_* : \Phi(J', p) \longrightarrow \Phi(A', p).$$

2.3.1 Teorema *One has*

$$\delta^{Dpq}(M) = \frac{\delta^D(pqM)}{c'_p c_q} \mathcal{E}(D, p, q, M)^2$$

where

$$\mathcal{E}(D, p, q, M) = |image(\xi_*)| \cdot |cokernel(\xi'_*)|.$$

Obviously

$$\text{Theorem 2} \Rightarrow \text{Assertion 1 of Theorem 1.}$$

2.4 Proof of Theorem 2

The proof of Theorem 2 relies on comparisons between the character groups of algebraic tori which are functorially associated to $J'_{/\mathbb{F}_p}$ and $J_{/\mathbb{F}_q}$.

General setting

If V is an abelian variety over $\mathbb{Q}$ and ℓ is a prime, let

$$T \quad = \quad \text{toric part of the fiber at } \ell \text{ of the Néron model for } V$$

and let $\mathcal{X}(V,\ell)$ be its character group:

$$\mathcal{X}(V,\ell) \quad = \quad Hom_{\overline{\mathbb{F}_\ell}}(T, \mathbb{G}_m).$$

Then

- $\mathcal{X}(V,\ell)$ is a free abelian group with compatible actions of: $\mathrm{Gal}(\overline{\mathbb{F}_\ell}/\mathbb{F}_\ell)$ and $End_{\mathbb{Q}}(V)$.
- If V has semistable reduction at ℓ then there is a canonical bilinear pairing (**monodromy pairing**), introduced by Grothendieck [3]:
  $$u_V : \mathcal{X}(V,\ell) \times \mathcal{X}(\check{V},\ell) \longrightarrow \mathbb{Z}$$
  giving rise to a natural exact sequence
  $$0 \to \mathcal{X}(V,\ell) \to \mathrm{Hom}(\mathcal{X}(V,\ell),\mathbb{Z}) \to \Phi(V,\ell) \to 0.$$

Steps for proving Theorem 2

Let $\delta = \delta^D(pqM)$ and $\delta' = \delta^{Dpq}(M)$.

- One reduces the claim to show that
  $$\frac{\delta' c'_p}{|coker\xi'_*|^2} = \frac{\delta c_q}{|coker\xi_*|^2}$$
- Let
  $$\begin{aligned} \mathcal{L} &= \text{the “f-part”of } \mathcal{X}(J,q) \\ \mathcal{L}' &= \text{the “f'-part”of } \mathcal{X}(J',p) \end{aligned}$$
  Then $\mathcal{L}$ (resp. $\mathcal{L}'$) is a no torsion subgroup of $\mathcal{X}(J,q)$ (resp. $\mathcal{X}(J',p)$) containing the image of $\xi^* : \mathcal{X}(A,q) \to \mathcal{X}(J,q)$ (resp. the image of $\xi'^* : \mathcal{X}(A',p) \to \mathcal{X}(J',p)$).

- Consider the diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{X}(A,q) & \to & \mathrm{Hom}(\mathcal{X}(A,q),\mathbb{Z}) & \to & \Phi(A,q) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{X}(J,q) & \to & \mathrm{Hom}(\mathcal{X}(J,q),\mathbb{Z}) & \to & \Phi(J,q) & \to & 0
\end{array}$$

  It is easy to show that

$$\begin{aligned}
|coker(\xi_*)| &= [\mathcal{L} : \mathcal{X}(A,q)] \\
|coker(\xi'_*)| &= [\mathcal{L}' : \mathcal{X}(A',p)]
\end{aligned}$$

  so that the claim reduces to

$$\frac{\delta' c'_p}{[\mathcal{L}' : \mathcal{X}(A',p)]} = \frac{\delta c_q}{[\mathcal{L} : \mathcal{X}(A,q)]}.$$

- By multiplicity 1, $\mathcal{L}$ (and $\mathcal{L}'$) have rank 1.

  Fix a generator g of $\mathcal{L}$ and a generator x of $\mathcal{X}(A,q)$.

- The maps

$$\begin{array}{lll}
\xi^* : \mathcal{X}(A,q) & \to \mathcal{X}(J,q) & \text{induced by } \xi \\
\xi_* : \mathcal{X}(J,q) & \to \mathcal{X}(A,q) & \text{induced by } \check{\xi}
\end{array}$$

  are self-adjoint w.r.t. monodromy, and $\xi^* \circ \xi_* = \delta$, so that

$$\begin{aligned}
\delta c_q &= \delta u_A(x,x) = u_A(x, \xi^*\xi_* x) = u_J(\xi^* x, \xi^* x) \\
&= [\mathcal{L} : \mathcal{X}(A,q)]^2 u_J(g,g)
\end{aligned}$$

  and analogously $\delta' c'_p = [\mathcal{L}' : \mathcal{X}(A',p)]^2 u_{J'}(g',g')$.

  Then the claim reduces to show that

$$u_J(g,g) = u_{J'}(g',g')$$

  where g is a generator of $\mathcal{L}$ and g' is a generator of $\mathcal{L}'$.

- We need to connect in some way g and g',

  Key point (Ribet [8] for $D = 1$, generalized by K. Buzzard [2]):

there is a canonical exact sequence

$$0 \to \mathcal{X}(J', p) \xrightarrow{i} \mathcal{X}(J, q) \to \mathcal{X}(J'', q) \times \mathcal{X}(J'', q) \to 0$$

where $J'' = J_0^D(qM)$.

The sequence is compatible with the Hecke action and monodromy pairing.

- then i embeds $\mathcal{L}'$ in $\mathcal{L}$, and $\mathcal{L}/i(\mathcal{L}')$ is torsion, but since $\mathcal{X}(J'', q)$ has no torsion, i restricts to an isomorphism $\mathcal{L}' \simeq \mathcal{L}$.
- Then we can pick $g = i(g')$ and the claim is proved.

2.5 Proof of Assertion 2

Let ℓ be a prime such that $A[\ell]$ is irreducible.

Then there exists an isogeny $A \to A'$ whose degree is not divisible by ℓ, so that

$$A[\ell] \simeq A'[\ell] \quad \text{as } G_{\mathbb{Q}} - \text{modules}.$$

and $ord_\ell(c_p) = ord_\ell(c'_p)$ for every prime p.

We define e as the ℓ-part of $\mathcal{E}$

$$e(D, p, q, M) = \ell^{ord_\ell \mathcal{E}(D,p,q,M)}.$$

Then $e(D, p, q, M) = e(D, q, p, M)$.

1 Proposition

$$e(D, p, q, M) = |coker(\xi'_* : \Phi(J', p) \to \Phi(A', p))|_\ell.$$

(Notice that q does not appear in the right hand)

PROVA:

By Theorem 2 this amounts to prove that the ℓ-part of $Im(\xi_* : \Phi(J, q) \to \Phi(A, q))$ is trivial.

FACT: $\Phi(J, q)$ is Eisenstein. (Ribet [8] for $D = 1$ and generalized to Shimura curve by Buzzard [2] and Jordan-Livné [4])

Then $Im(\xi_*)$ is annihilated by $a_r(f) - r - 1$ for every prime r.

$\Rightarrow$ its ℓ-part is trivial, otherwise $a_r \equiv r + 1 \mod \ell$ for every prime r

which is a contradiction, because $A[\ell]$ is irreducible. $\square$

Then we can consider varying decompositions $N = DpqM$.

Put $M = M'rs$. (By hypothesis M' is square-free but not prime!).

Then

$$e(Drs, p, q, M') = e(Dqs, p, r, M')$$

because each one is the order of the ℓ-part of

$$cocker(\xi'_* : \Phi(J^{Dpqrs}(M'), p) \to \Phi(A', p)).$$

By Assertion 1

$$\left(\frac{\delta^{pqrsD}(M')c_p c_q c_r c_s}{\delta^D(pqrsM')}\right)_\ell = \begin{array}{c} e(Drs, p, q, M')^2 e(D, r, s, M'pq)^2 \\ \| \\ e(Dqs, p, r, M')^2 e(D, q, s, M'pr)^2 \end{array}$$

so that $e(D, r, s, M'pq) = e(D, q, s, M'pr)$.

In conclusion, it follows that if ℓ divides $e(D, p, q, M)$ then ℓ divides c_r for every prime r dividing $N = DpqM$.

By a previous result of Ribet [8], then f should be congruent modulo ℓ to a form in $S_2(\mathrm{SL}_2(\mathbb{Z}))$. But $S_2(\mathrm{SL}_2(\mathbb{Z}))$ is zero; therefore $e(D, p, q, M) = 1$.

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