Overview of the Patching Argument

Josu Pérez Zarraonandia

Abstract

Let $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{Q}_p)$ be a Galois representation, and assume that there exists another Galois representation r which arises from an automorphic form, and is such that $\overline{\rho} \simeq \overline{r} \pmod{p}$. Can we deduce then that ρ arises from an automorphic form as well? The patching argument, pioneered by Wiles and Taylor, allows us to make this deduction.

Let $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{Q}_p)$ be a Galois representation, and assume that there exists another Galois representation r which arises from an automorphic form, and is such that $\overline{\rho} \simeq \overline{r} \pmod{p}$. Can we then deduce that ρ arises from an automorphic form as well? To answer this question we will take another perspective on our problem, so let us rephrase our question in the following way. Let $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{F}_p)$ be the residual representation attached to ρ. Can we identify which of the representations lifting $\overline{\rho}$ are automorphic?

The advantage of this point of view is the introduction of techniques from deformation theory in the study of our problem. In fact, via Mazur's deformation theory, we are led to a universal deformation ring $R_{\overline{\rho}}$ which parametrizes all lifts of $\overline{\rho}$, meaning that there exists a universal Galois representation $\rho^{\mathrm{univ}}\colon G_{\mathbb{Q}} \to \mathrm{GL}_n(R_{\overline{\rho}})$ lifting $\overline{\rho}$, and such that any other lift with coefficients in a ring S arises as a composition of ρ^{univ} and a map $R_{\overline{\rho}} \to S$.

In particular, this means that all automorphic representations factor through a map $R_{\overline{\rho}} \to \mathbb{T}$, where $\mathbb{T}$ denotes the Hecke algebra. To understand where the patching argument comes from, we need to look at $\mathrm{Spec}(\mathbb{T})$ as a closed subspace of $\mathrm{Spec}(R_{\overline{\rho}})$. A careful study of the irreducible components of $\mathrm{Spec}(\mathbb{T})$ inside $\mathrm{Spec}(R_{\overline{\rho}})$ reveals an argument which can give a positive answer to our original question. However, the singularities of $\mathrm{Spec}(R_{\overline{\rho}})$ stop our naive argument from working, and we require more sophisticated methods to work around this obstacle.

In an attempt to smoothen the singularities of the universal deformation ring, Wiles and Taylor proposed patching together a family of auxiliary universal rings that arise when modifying slightly our deformation problem. After this process, we obtain an abstract ring on which we can perform computations. In particular, we can prove certain facts about the irreducible components of this ring which can then be transferred back to our original ring, allowing us to answer our question positively.

References

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor. Automorphy for some ℓ-adic lifts of automorphic mod ℓ Galois representations. *Publications Mathématiques de l'IHÉS*, **108** (2008) 1–181.

[Gee13] Toby Gee. Modularity Lifting Theorems. `https://www.ma.imperial.ac.uk/~tsg/Index_files/ArizonaWinterSchool2013.pdf`, 2013. Accesed: 2025-05-17.

[Maz97] Barry Mazur. *An Introduction to the Deformation Theory of Galois Representations*, pages 243–311. Springer New York, New York, NY, 1997. ISBN 978-1-4612-1974-3.

[TW95] Richard Taylor and Andrew Wiles. Ring-theoretic properties of certain Hecke algebras. *Annals of Mathematics*, **141** (1995) 553–572.

[Wil95] Andrew Wiles. Modular elliptic curves and Fermat's Last Theorem. *Annals of Mathematics*, **141** (1995) 443–551.