Formal confluence of quantum differential operators

Adolfo Quirós
(Joint work with Bernard Le Stum)

Universidad Autónoma de Madrid
(Université de Rennes, France)

30STNB, Barcelona, 26 January 2016

«Grothendieck-Berthelot» differential operators

Cálculus on rings

R =ring, $P = \sum a_i t^i \in R[t]$, $\partial := \frac{d}{dt} \Rightarrow \partial(P) = \sum i a_i t^{i-1}$.

When $R = \mathbb{Z}, \mathbb{Z}_p, \mathbb{F}_p, \mathbb{F}_q$ some elementary results do not hold.

Examples:

- Taylor formula: $P = \sum \partial^i P(0) \frac{t^i}{i!}$.
- Poincaré lemma: $\int t^{p-1} dt = \frac{t^p}{p}$
- In $\mathbb{F}_p$ [characteristic $p > 0$], what is the order of ∂^p?

  $\partial^p(PQ) = \sum_{i=0}^{p} \binom{p}{i} \partial^i(P) \partial^{p-i}(Q) = P\partial^p(Q) + \partial^p(P)Q$.

  ∂^p is a derivation= a differential operator of order 1.

Poincaré Lemma over $\mathbb{C}$

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}[t] \xrightarrow{\partial} \mathbb{C}[t] \longrightarrow 0 \qquad \text{is exact.}$$

Grothendieck's idea (Berthelot)

There is no Poincaré Lemma when $R = \mathbb{F}_p, \mathbb{Z}_p$.

$$0 \longrightarrow R \longrightarrow R[t] \xrightarrow{\partial} R[t] \longrightarrow 0$$

- ∂ is not onto: $\partial(x) = t^{p-1} \Rightarrow x = \int t^{p-1} dt = \frac{t^p}{p}$.
- When $R = \mathbb{F}_p$, $\partial(t^p) = 0 \Rightarrow Ker(\partial) \neq \mathbb{F}_p$.

Assume we add Divided Powers:

elements $t^{[i]}$ that behave $\sim \frac{t^i}{i!}$.

In particular $\partial(t^{[i]}) = t^{[i-1]}$.

Let $R < t > := R[t^{[i]} \mid i \geq 0]$, then

Poincaré Lemma with Divided Powers

$$0 \longrightarrow R \longrightarrow R < t > \xrightarrow{\partial} R < t > \longrightarrow 0$$ is exact.

Divided Powers (DP / PD)

Definition. A a ring, $I \subset A$ an ideal.

Divided Powers on I means, for $x \in I$, elements $A \ni x^{[i]} \sim \frac{x^i}{i!}$:

- $x^{[0]} = 1, x^{[1]} = x$.
- $(x+y)^{[n]} = \sum_{i=0}^{n} x^{[i]} y^{[n-i]}$.
- $(ax)^{[i]} = a^i x^{[i]}$.
- $x^{[i]} x^{[j]} = \binom{i+j}{i} x^{[i+j]} \qquad [\binom{i+j}{i} \in \mathbb{Z}]$.
- $(x^{[i]})^{[j]} = \frac{(ij)!}{j!(i!)^j} x^{[i+j]} \qquad [\frac{(ij)!}{j!(i!)^j} \in \mathbb{Z}]$.

Examples of PD-ideals

- $(p) \subset \mathbb{Z}_p$ has PD: $p^{[i]} = p^i / i! \in \mathbb{Z}_p$.
- $(\sqrt[p]{p}) \subset \mathbb{Z}_p[\sqrt[p]{p}]$ does NOT have PD: $\mathrm{ord}_p(\sqrt[p]{p}^{p^2} / p^2!) = -1$.
- $\mathbb{F}_p < t >, \mathbb{Z}_p < t >, I = (t^{[i]} \mid i \geq 0)$ has PD.
- $\mathbb{F}_p[t], \mathbb{Z}_p[t], I = (t)$ does NOT have: $p! t^{[p]} = t^p \Rightarrow t^{[p]} \notin (t)$.

Alternative idea

Put divided powes on the operators.

[local, one variable.] R ring, $X := \mathbb{A}^1 = \operatorname{Spec} R[t]$, $\partial := \frac{d}{dt}$.

Differential operators of level 0 on $\mathbb{A}^1$.

They are the non commutative $R[t]$-algebra $\mathcal{D}^{(0)}$ generated by ∂ with the usual commutation rules: $\partial \cdot P - P \cdot \partial = \partial(P)$.

Note (Berthelot): We can build $\mathcal{D}^{(0)}$ using the «EGA construction» we will see later, but using divided powers $I^{[n+1]}$ in stead of I^{n+1}. This is the «level 0».

Differential operators of higher level (Berthelot)

But there are also differential operators «with PD»: $\partial^{[i]} \sim \frac{\partial^i}{i!}$.

- $\partial^{[i]}(t^j) = \binom{j}{i} t^{j-i} \in R[t]$.
- $R = \mathbb{Z}, \mathbb{Z}_p, \mathbb{F}_p \Rightarrow \partial^{[p]} \notin \mathcal{D}^{(0)}$.
- $R = \mathbb{Z}, \mathbb{Z}_p, \mathbb{F}_p \Rightarrow \partial^{[p]}$ is a diff. op. of order p [∂^p has ord. 1].

Differential operators of level m on $\mathbb{A}^1$.

They are the non commutative $R[t]$-algebra $\mathcal{D}^{(m)}$ generated by $\{\partial^{[p^l]} | l \leq m\}$ with the natural commutation rules.

Differential operators of level ∞ on $\mathbb{A}^1$.

They are the non commutative $R[t]$-algebra $\mathcal{D}^{(\infty)}$ generated by $\{\partial^{[p^l]} | l \geq 0\}$ with the natural commutation rules.

$\mathcal{D}^{(\infty)}$ gives a Taylor formula:

$$P \in R[t], \quad P = \sum \partial^{[i]} P(0) t^i.$$

Grothendieck's construction [EGA IV]

(Valid on a scheme: change $A \rightsquigarrow \mathcal{O}_X$)

- A a commutative R-algebra.
- $P_{A/R} := A \otimes_R A$ with the *left* A-module structure. For $x \in A$, we write in $P_{A/R}$:
  - $x = x(1 \otimes 1) = x \otimes 1$.
  - $\tilde{x} = 1 \otimes x$.
  - NOTE: $P_{A/R} = \{\sum x_i \tilde{y}_i\}$.
  - We call Taylor map the embedding on the right

$$\theta_{A/R} : \quad \begin{array}{ccc} A & \longrightarrow & P_{A/R}. \\ x & \longmapsto & \tilde{x} \end{array}$$

- $I_{A/R}$ be the kernel of the multiplication morphism

$$\begin{array}{ccc} P_{A/R} & \longrightarrow & A \\ x\tilde{y} & \longmapsto & xy \end{array}$$

  It is generated by the image of the linear map

$$d_{A/R} : \quad \begin{array}{ccc} A & \longrightarrow & P_{A/R} \\ x & \longmapsto & \xi := \tilde{x} - x \end{array}$$

Principal parts and differential operators

A-module of principal parts of order $n \in \mathbb{N}$ (and ∞-level) of A:

$$P_{A/R,(n)} := P_{A/R}/I_{A/R}^{n+1}.$$

The A-module of principal parts of infinite order (and ∞-level) of A is the *completion*:

$$\widehat{P}_{A/R} := \varprojlim P_{A/R,(n)}.$$

A-module of differential operators of order <u>at most</u> $n \in \mathbb{N}$ (and ∞-level) of A:

$$\mathcal{D}^{(\infty)}_{A/R,(n)} := \mathrm{Hom}_A(P_{A/R,(n)}, A).$$

A-module of differential operators (of ∞-level) of A:

$$\mathcal{D}^{(\infty)}_{A/R} := \cup \mathcal{D}^{(\infty)}_{A/R,(n)} = \varinjlim \mathrm{Hom}_A(P_{A/R,(n)}, A).$$

Explicit formulas for $A = R[x]$

- Recall $\xi = \tilde{x} - x = 1 \otimes x - x \otimes 1$.
- We say x is a coordinate for A over R if for all $n \in \mathbb{N}$, $P_{A/R,(n)}$ is freely generated as an A-module by the images of $1, \xi, \xi^2, \ldots, \xi^n$.
- Example: x is a coordinate for $A = R[x]$.

Dual basis: «PD on differential operators»

Let x be a coordinate on A. Then, $\{\partial^{[k]}\}$ is the basis of $\mathcal{D}^{(\infty)}_{A/R,(n)}$ dual to the basis $\{\xi^{(k)}\}$ of $P_{A/R,(n)}$. It is characterized by

$$\forall k, n \in \mathbb{N}, \quad \partial^{[k]}(\xi^{(n)}) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}.$$

$$\mathcal{D}^{(\infty)}_{A/R} = \{\sum_{finite} a_k \partial^{[k]} : a_k \in A\}.$$

Twisted differential operators

Review of Quantum arithmetic

Let R be a conmutative ring, $q \in R$, $n, k \in \mathbb{N}$.

Quantum (states of) numbers and quantum factorials

$$(n)_q := \sum_{i=0}^{n-1} q^i, \qquad (n)_q! := \prod_{i=1}^{n} (i)_q.$$

Quantum binomial coefficients [Definition by induction]

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q,$$

with

$$\binom{0}{k}_q = \begin{cases} 1 \text{ if } k = 0 \\ 0 \text{ otherwise} \end{cases}$$

If all q-integers are invertible in R

$$\binom{n}{k}_q = \frac{(n)_q!}{(k)_q!(n-k)_q!}.$$

Quantum characteristic

Definition

1. $q - char(R) = min\{m : (m)_q = 0\}$ or 0.
2. R is *q-flat* if it has no $(m)_q$-torsion for any $m \in \mathbb{N}$ $(m \neq 0)$.
3. R is *q-divisible* if $(m)_q$ is invertible whenever $(m)_q \neq 0$.

Examples

1. $q = 1$: $q - char(R) = char(R)$ and
   - R is q-flat if either it has no $\mathbb{Z}$-torsion ($char(R) = 0$) or it is an $\mathbb{F}_p$-algebra ($char(R) = p$ prime).
   - R is q-divisible $\iff$ it is a K-algebra (K a field).
2. $R = A[t]$, A a ring, $q = t$: R is q-flat, but not q-divisible, and $q - char(R) = 0$ (for any A).
3. $R = \mathbb{C}, q = e^{\frac{2\pi\sqrt{-1}}{p}}$: R is q-divisible and $q - char(R) = p$.

Twisted derivations

Definition

Let R be a conmutative ring

1. A *twisted R-algebra* is a commutative R-algebra, A, with an R-algebra endomorphism $\sigma : A \longrightarrow A$.
2. A *twisted derivation* of A is a $D \in Hom_R(A, A)$ such that

$$\forall x, y \in A \qquad D(xy) = yD(x) + \sigma(x)D(y).$$

NOTE: $r \in R \Rightarrow D(r) = 0$.

Notation

$$Der_{R,\sigma}(A, A) =: T_{A/R,\sigma}$$

The ring of small twisted differential operators

Definition

The *ring of small twisted differential operators* $\overline{D}_{A/R,\sigma}$ is the subring of $End_R(A)$ generated by A and $T_{A/R,\sigma}$.

Definition

$x \in A$ is a *twisted coordinate* for A if

$$\begin{aligned} T_{A/R,\sigma} &\longrightarrow A \\ D &\longmapsto D(x) \end{aligned}$$

is a bijection.

The unique twisted derivation such that $\partial_{x,\sigma}(x) = 1$ is given by:

$$\forall n \in \mathbb{N}, \quad \partial_{x,\sigma}(x^n) = \sum_{i=0}^{n-1} x^i \sigma(x)^{n-1-i}.$$

Examples of twisted coordinates and derivations

1. A/R smooth, $\sigma = \mathrm{Id_A}$: x is a twisted coordinate $\Longleftrightarrow$ the map $R[X] \to A$ that sends X to x is étale. $\partial_\sigma = \partial_x$.
2. $A = R[x]$: x is a twisted coordinate for any σ. For instance:
   - For $\sigma(x) = qx, q \in R : \partial_\sigma(x^n) = (n)_q x^{n-1}$.
3. $A = R[x, x^{-1}], \sigma(x) = qx, q \in R^*$: x is also a coordinate,
$$\partial_\sigma(x^n) = (n)_q x^{n-1}, \qquad n \in \mathbb{Z}.$$
4. $R = K$ a field, $A = K(x)$, σ an endomorphism of $K(x)/K$: ∂_σ is given by the general formulas and
$$\partial_\sigma\left(\frac{y}{z}\right) = \frac{\partial_\sigma(y)z - y\partial_\sigma(z)}{z\sigma(z)}.$$

The twisted Weyl algebra

Definition

The *twisted Weyl algebra over A* is the non commutative A-algebra $D_{A/R,\sigma}$ on one generator ∂_σ with relations:

$$\forall x \in A, \qquad \partial_\sigma x = \partial_{A,\sigma}(x) + \sigma_A(x)\partial_\sigma.$$

Proposition

Let x be a twisted coordinate on A. Then there is a canonical surjective morphism

$$D_{A/R,\sigma} \twoheadrightarrow \overline{D}_{A/R,\sigma}\ .$$

If $\sigma(x) = qx$, $q - char(R) = 0$ and A is q-flat, $D_{A/R,\sigma} = \overline{D}_{A/R,\sigma}$, but this is not true in general.

Examples

1. $R = \mathbb{C}, A = \mathbb{C}[x], \sigma = id_A$, or $\sigma(x) = x + h$, or $\sigma(x) = qx$, with q not a root of unity. We have
$$D_{A/R,\sigma} = \overline{D}_{A/R,\sigma}.$$

2. $R = \mathbb{F}_2, A = \mathbb{F}_2[x], \sigma = id_A$. Then
$$char(A) = 2 \Rightarrow \partial^2 = 0 \in End_R(A).$$
Hence $\overline{D}_{A/R,\sigma} = A \oplus T_{A/R,\sigma} \neq D_{A/R,\sigma}$.

3. $R = \mathbb{C}, A = \mathbb{C}[x], \sigma(x) = -x$.
We also have $\overline{D}_{A/R,\sigma} = A \oplus T_{A/R,\sigma} \neq D_{A/R,\sigma}$.
We are in $char(A) = 0$, but $q - char(R) = 2$.

A look at the classics

1. When x is a twisted coordinate such that $\sigma(x) = x + h$:
   A is a *finite difference algebra* and $\partial_\sigma = \Delta_h$ where

$$\Delta_h(f)(x) = \frac{f(x+h) - f(x)}{h}.$$

2. When x is a twisted coordinate such that $\sigma(x) = qx$:
   A is a *q-difference algebra* and $\partial_\sigma = \delta_q$ where (assuming $q - 1 \in R^*$ and $x \in A^*$)

$$\delta_q(f)(x) = \frac{f(qx) - f(x)}{qx - x}.$$

3. When $\sigma = id_A$ and x is a (usual) coordinate:
   A is a *(usual) differential algebra* and $\partial_\sigma = \partial$. Moreover, whenever it has a meaning,

$$\partial(f)(x) = lim_{h \to 0} \Delta_h(f)(x) = lim_{q \to 1} \delta_q(f)(x).$$

Twisted diff. operators à la Grothendieck-Bethelot

- We want to see a ring of twisted differential operators as some dual of a ring of formal functions.
- Doing this directly for the twisted Weyl Algebra D_σ would require to understand the notion of σ-divided powers on the function side. Doable, but not easy.
- We will replace D_σ with a Grothendieck ring of differential operators $\mathrm{D}_\sigma^{(\infty)}$, so that the σ-divided powers live naturally on the differential operator side. The classical construction works incredibly well for this particular generalization.

Twisted principal parts, twisted diff. operators

- Twisted powers of I:

$$I^{(0)} = P, \quad I^{(1)} = I, \quad I^{(2)} := I\sigma(I), \quad \dots, \quad I^{(n)} := I\sigma(I)\cdots\sigma^{n-1}(I).$$

- A-module of twisted principal parts of order $n \in \mathbb{N}$ (and ∞-level) of A:

$$P_{A/R,(n)_\sigma} := P_{A/R}/I_{A/R}^{(n+1)_\sigma}.$$

- The A-module of twisted principal parts of infinite order (and ∞-level) of A is the *completion*:

$$\widehat{P}_{A/R_\sigma} := \varprojlim P_{A/R,(n)_\sigma}.$$

- A-module of twisted differential operators of order at most $n \in \mathbb{N}$ (and ∞-level) of A:

$$\mathrm{D}^{(\infty)}_{A/R,(n)_\sigma} := \mathrm{Hom}_A(P_{A/R,(n)_\sigma}, A).$$

  A-module of twisted differential operators (of ∞-level) of A:

$$\mathrm{D}^{(\infty)}_\sigma := \cup \mathrm{D}^{(\infty)}_{A/R,(n)_\sigma} = \varinjlim \mathrm{Hom}_A(P_{A/R,(n)}, A).$$

Quantum confluence

Quantum diferential operators of infinite level

Let A be a quantum R-algebra with a twisted coordinate x such that $\sigma(x) = qx + h$ with $q, h \in R$.

With a «Grothendieck-Berthelot construction» we can define for all k twisted differential operators (of order k), $\partial_\sigma^{[k]}$.

They are characterized by

$$\forall k, l \in \mathbb{N}, \quad \partial_\sigma^{[k]} \circ \partial_\sigma^{[l]} = \binom{k+l}{l}_q \partial_\sigma^{[k+l]}.$$

From which we deduce

$$\forall k \in \mathbb{N}, \forall z \in A, \quad \partial_\sigma^{k}(z) = (k)_q! \partial_\sigma^{[k]}(z).$$

The ring of quantum diferential operators of infinite level is the A-algebra $D^{(\infty)}_{A/R,\sigma}$ generated by the operators $\partial_\sigma^{[k]}, k \in \mathbb{N}$.

The key comparison theorem

We have an epi-mono factorization

$$D_{A/R,\sigma} \twoheadrightarrow \overline{D}_{A/R,\sigma} \hookrightarrow D^{(\infty)}_{A/R,\sigma}.$$

Moreover, if R is q-divisible, then

1. If $q - char(A) = 0$, we have
$$D_{A/R,\sigma} = \overline{D}_{A/R,\sigma} = D^{(\infty)}_{A/R,\sigma}.$$
2. If $q - char(A) = p > 0$, we have
$$D_{A/R,\sigma}/\partial_\sigma^p \simeq \overline{D}_{A/R,\sigma} \simeq D^{(\infty)}_{A/R,\sigma}/K_\sigma^{[p]},$$
where $K_\sigma^{[p]}$ is the free A-module generated by all $\partial_\sigma^{[k]}$ with $k \geq p$ (it is a left ideal).

«Proof»

$$\partial_\sigma^k \longmapsto (k)_q!\partial_\sigma^{[k]}.$$

The twist does not matter (in the limit)

Let $K_\sigma^{[m]}$ be the free A-module (left ideal) generated by all $\partial_\sigma^{[k]}$, $k \geq m$. They define the (decreasing) ideal filtration on $\mathrm{D}_\sigma^{(\infty)}$.

The module of twisted differential operators of infinite level and infinite order on A is

$$\widehat{\mathrm{D}}_{A/R,\sigma}^{(\infty)} = \varprojlim \mathrm{D}_\sigma^{(\infty)} / K_\sigma^{[m+1]}.$$

It is not a ring in general since multiplication is not continuous with respect to the filtration: $\partial_\sigma^{[k]} \circ x = \sigma^k(x)\partial_\sigma^{[k]} + \partial_\sigma^{[k-1]}$.

Formal deformation theorem

If x is a also a τ-coordinate, then there exists an isomorphism of topological A-modules (i. e. compatible with the ideal filtrations) that depends only on x

$$\widehat{\mathrm{D}}_{A/R,\sigma}^{(\infty)} \xrightarrow{\simeq} \widehat{\mathrm{D}}_{A/R,\tau}^{(\infty)} .$$

Enter the classical case

The Weyl algebra $D_{A/R,\sigma}$ can also be filtered through the ideals generated by ∂_σ^k, and the composite map

$$D_{A/R,\sigma} \twoheadrightarrow \overline{\mathrm{D}}_{A/R,\sigma} \hookrightarrow \mathrm{D}^{(\infty)}_{A/R,\sigma}$$

is then compatible with the ideal filtrations.

Suppose $q - char(R) = 0$ and R is q-divisible.

- It follows from the key comparison theorem that

$$\widehat{\mathrm{D}}_{A/R,\sigma} \simeq \widehat{\mathrm{D}}^{(\infty)}_{A/R,\sigma}.$$

- This applies in particular when R is a $\mathbb{Q}$-algebra and $\sigma = \mathrm{Id}$: the «classical case».
- If x is also a usual coordinate (that is, for $\tau = id$), formal deformation gives

$$\mathrm{D}_\sigma \to \mathrm{D}^{(\infty)}_\sigma \hookrightarrow \widehat{\mathrm{D}}^{(\infty)}_\sigma \simeq \widehat{\mathrm{D}}^{(\infty)} \simeq \widehat{\mathrm{D}}_{A/R}.$$

Formal quantum confluence theorem

Let R be a $\mathbb{Q}$-algebra, A a q-R-algebra for some $q \in R$. Assume that the quantum coordinate on A is also a usual coordinate. Then, there exists a canonical map of filtered A-modules

$$\mathrm{D}_{A/R,\sigma} \to \widehat{\mathrm{D}}_{A/R}.$$

Moreover, if R is q-divisible, we have

1. If $q\mathrm{-char}(A) = 0$, then the map is injective and $\mathrm{D}_{A/R,\sigma}$ is dense in $\widehat{\mathrm{D}}_{A/R}$.
2. If $q\mathrm{-char}(A) = p > 0$, then the map induces an isomorphism

$$\left(\mathrm{D}_{A/R,\sigma}/\partial_\sigma^p \simeq\right) \quad \overline{\mathrm{D}}_{A/R,\sigma} \simeq \mathrm{D}_{A/R}/\partial^p.$$

A q-characteristic 0 example

- R a field of characteristic zero and q not a root of unity, i. e. q-characteristic $= 0$
- $\tau(x) = x$, $\partial_\tau =: \partial$.
- $\sigma(x) = qx$, $\partial_\sigma =: \partial_q$.
- Then:

$$\partial_q = \sum_{k=1}^{\infty} \frac{1}{k!}(q-1)^{k-1} x^{k-1} \partial^k$$

$$\partial = \sum_{k=1}^{\infty} \frac{(1-q)^k}{1-q^k} x^{k-1} \partial_q^k.$$

Quantum confluence in positive q-characteristic

(Strongly) admissible system of roots

- A system of roots of $q \in R$ is a compatible family $\underline{q} := \{q_n\}_{n \in S}$ of n-th roots of q.
- We are particularly interested in non trivial roots of unity.
- We call the system $\underline{q}$ admissible if $(n)_{q^n} \in R^\times, \forall n \in S$.
- When $\underline{q}$ is admissible we can define the *q-rational number* $(r)_{\underline{q}}$ *for* any $r \in \mathbb{N}\frac{1}{S}$: if $r = \frac{m}{n}$ with $m \in \mathbb{N}$ and $n \in S$, then,

$$(r)_{\underline{q}} := \frac{(m)_{q_n}}{(n)_{q_n}}$$

  only depends on r and not on the choice of m and n.
- We say $\underline{q}$ is strongly admissible if $(n)_{q_n}! \in R^\times$, for all $n \in S$.

Divisible systems of roots

- If $\underline{q}$ is an admissible (resp. a strongly admissible) system of roots, and we write for all $n \in S$, $p_n := q_n\mathrm{-char}(R)$, then we have
$$\forall n \in S, \quad p_n = 0 \quad \text{or} \quad p_n \nmid n \text{ (resp. } p_n > n).$$
If R is $\underline{q}$-divisible (q_n-divisible for all $n \in S$) the converse is true.
- If we are given an *admissible* system $\underline{q}$ of roots of $q \in R$ and some $h \in R$, we can endow the R-algebra A with a system $\underline{\sigma} := (\sigma_n)_{n\in S}$ of endomorphisms:
$$\sigma_n(x) = q_n x + \left(\frac{1}{n}\right)_q h.$$
- Assume that R is $\underline{q}$-divisible, $q\mathrm{-char}(R) = p > 0$ and $\underline{q}$ is a system of p^n-th roots of q. Then $q_{p^n}\mathrm{-char}(R) = p^{n+1} > p^n$ and hence the system is strongly admissible.

Confluence in positive quantum characteristic

If A is a $\underline{q}$-R-algebra with $\underline{q}$ strongly admissible, then the rooted twisted Weyl algebra of A is

$$\mathrm{D}_{A/R,\underline{\sigma}} := \varinjlim \mathrm{D}_{A/R,\sigma_n}.$$

Theorem: «untwist»

Let R be a $\mathbb{Q}$-algebra, $\underline{q}$ a strongly admissible system of roots in R and A a $\underline{q}$-R-algebra.

1. There exists a canonical A-linear map

$$\mathrm{D}_{A/R,\underline{\sigma}} \to \widehat{\mathrm{D}}_{A/R}.$$

Construction

Glue together the morphisms $\mathrm{D}_{A/R,\sigma_n} \to \widehat{\mathrm{D}}_{A/R}$ given by formal quantum confluence.

Going to the limit

Theorem (continued): the confluence

2. If R is $\underline{q}$-divisible and $\underline{q}$ is infinite, then the map $\mathrm{D}_{A/R,\underline{\sigma}} \to \widehat{\mathrm{D}}_{A/R}$ has dense image

PROOF:

For all $n \in S$, set $p_n := q_n - \mathrm{char}(R)$ and apply the formal quantum confluence theorem to σ_n. If all $p_n = 0$, we are done. When $p_n \neq 0$, the bottom map in the following commutative diagram is surjective:

$$\begin{array}{ccc} \mathrm{D}_{\underline{\sigma}} & \longrightarrow & \widehat{\mathrm{D}} \\ \uparrow & & \downarrow \\ \mathrm{D}_{\sigma_n} & \twoheadrightarrow & \mathrm{D}/\partial^{p_n} \end{array} .$$

Since $\underline{q}$ is infinite, $p_n > n$ for infinitely many $n's$. Hence $p_n \to \infty$ and the image of the upper map is dense.

What are we after?

Done (but not yet in the ArXiv)

Quantum differential operators + Divided Powers $\Rightarrow$ Azumaya nature of the algebra of quantum differential operators $\Rightarrow$ Quantum Simpson correspondence: a quantum version of «there exists an equivalence between quasi-nilpotent Higgs bundles on $X^{(m+1)}$ and quasi-nilpotent $\mathcal{D}^{(m)}_{X/S}$-modules on X».

Goal

To build (understand?) the *p*-adic Simpson correspondences as a confluence of quantum correspondences.

Thank you!

Moltes gràcies!

¡Muchas gracias!