p-adic interpolation of modular forms of infinite slope

Seminari de Teoria de Nombres

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Modular forms

A modular form f is a holomorphic function on the complex upper half-plane

$$\mathcal{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$$

that satisfies a *weight* $k \geq 1$ transformation property with respect to a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ of *level* $N \geq 1$:

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for every $z \in \mathcal{H}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, plus a holomorphy condition at cusps.

Our Γ will always be $\Gamma_0(N), \Gamma_1(N)$, or some combination of the two. We can expand f as a Fourier series in $q = e^{2\pi iz}$:

$$f = \sum_{i \geq 0} a_i q^i$$

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Eigenforms

The space of modular forms of fixed level and weight admits an action of an algebra of *Hecke operators* : an operator T_ℓ for every prime $\ell \nmid N$, and operators U_p for primes $p \mid N$.

If f is an eigenvector for the Hecke action (=an *eigenform*) and $a_1 = 1$, then a_ℓ (for $p \mid N$, a_p) is the Hecke eigenvalue for the operator T_ℓ (for $p \mid N$, U_p).

For every field K, write $G_K = \mathrm{Gal}(\overline{K}/K)$.

For every prime p, we can "attach" to f a 2-dimensional $\mathbb{Q}_p$-vector space with a continuous action

$$\rho_{f,p} \colon G_{\mathbb{Q}} \to \mathrm{GL}(V),$$

that satisfies, for every prime $\ell \nmid Np$:

- $\rho_{f,p}$ is unramified at ℓ,
- the trace of $\rho_{f,p}(\mathrm{Frob}_\ell)$ is a_ℓ.

(Eichler–Shimura, Deligne, Deligne–Serre)

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From now on, fix a prime p and a natural number N with $p \nmid N$.

Definition

A p-adic family of eigenforms of tame level N is a power series

$$\mathcal{F}(T, q) = \sum_{i \geq 0} a_i(T) q^i \in \overline{\mathbb{Z}}_p[[T, q]]$$

such that, for every $k \geq 2$, the specialization of $\mathcal{F}$ at the arithmetic prime

$$w_k = (1 + T) - (1 + p)^k$$

is the q-expansion of a modular eigenform f_k of weight k and level Np^r for some $r \geq 1$.

To a family as above we can attach a continuous representation

$$G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Z}}_p[[T]])$$

that specializes to $\rho_{f_k, p}$ at w_k. In particular, every point of the family carries a representation of $G_{\mathbb{Q}}$.

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An explicit example

The *slope* of a modular form is the p-adic valuation of a_p. The slope is finite iff $a_p \neq 0$.

Let ψ be a Grössencharacter of an imaginary quadratic field K, of infinity type $(k-1, 0)$. Then

$$\sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{N_{K/\mathbb{Q}}(\mathfrak{a})},$$

where the sum is over fractional ideals of K prime to the conductor of ψ and $N_{K/\mathbb{Q}}$ denotes the norm, is the q-expansion of an eigenform f of weight k. We say that f is a CM form (for *complex multiplication*). The slope of f can be infinite !

The associated $\rho_{f,p}$ is induced by a character $G_K \to \overline{\mathbb{Q}}_p^\times$.

By explicitly varying ψ, we get a family $\mathcal{F}$ of CM eigenforms (Hida). The associated $\rho_{\mathcal{F}}$ is induced by a character $G_K \to \overline{\mathbb{Z}}_p[[T]]^\times$.

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Twisting

Given any form $f = \sum_{i \geq 1} a_i q^i$ (or family) and a Dirichlet character δ of conductor M, we can *twist* the form (or family) by δ :

$$\delta f = \sum_{i \geq 0} \delta(i) a_i q^i.$$

Then $\rho_{\delta f, p} = \rho_{f,p} \otimes \delta$.

If $p \mid M$, the slope of δf is infinite.

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Existence of families

Theorem (Hida, Coleman–Mazur, Buzzard, Chenevier)

Every eigenform of finite slope (i.e. with $a_p \neq 0$) lives in a p-adic family.

Question (Coleman–Mazur) : what about eigenforms of infinite slope ?

For families of infinite slope to have some meaning, we have to replace the interpolation of a_p with that of the eigenvalues of the "semistable Frobenius".

Theorem? (C.)

All families of eigenforms of infinite slope are either CM, or twists of families of finite slope.

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p-supercuspidal forms

Let π_p be the admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to f by the local Langlands correspondence. Let M be the level of f.

If the slope of f is infinite, then

1. either f is a twist of an eigenform of finite slope by a Dirichlet character,
2. or $p^2 \mid M$ and π_p is a supercuspidal representation.

We can rephrase the main theorem : all families of p-supercuspidal forms are CM.

It is enough to prove that the associated Galois representation is induced.

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p-adic Hodge theory

How to characterize the representations attached to (p-adic families) of eigenforms among all continuous representations of $G_{\mathbb{Q}}$?

Use p-adic Hodge theory : recall Fontaine's rings of periods

$$\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$$

$$B_{\mathrm{dR}}^+ \cong \mathbb{C}_p[[t]], B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[1/t]$$

$$B_{\mathrm{cris}},\ B_e = B_{\mathrm{cris}}^{\varphi=1}$$

All are equipped with actions of $G_{\mathbb{Q}_p}$.

Definition

A B-pair is a pair (W_e, W_{dR}^+) consisting of a B_e-semilinear representation W_e of $G_{\mathbb{Q}_p}$ and a $G_{\mathbb{Q}_p}$-stable lattice W_{dR}^+ inside of $W_e \otimes_{B_e} B_{\mathrm{dR}}$.

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A B-pair is a pair (W_e, W_{dR}^+) consisting of a B_e-semilinear representation W_e of $G_{\mathbb{Q}_p}$ and a $G_{\mathbb{Q}_p}$-stable lattice W_{dR}^+ inside of $W_e \otimes_{B_e} B_{\mathrm{dR}}$.

p-adic Hodge theory

How to characterize the representations attached to (p-adic families) of eigenforms among all continuous representations of $G_{\mathbb{Q}}$?

Use p-adic Hodge theory : recall Fontaine's rings of periods

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$$B_{\mathrm{dR}}^+ \cong \mathbb{C}_p[[t]], B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[1/t]$$

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Triangulline representations

To a continuous $\mathbb{Q}_p$-linear representation V of $G_{\mathbb{Q}_p}$ we attach the B-pair

$$W(V) = (V \otimes_{\mathbb{Q}_p} B_e, V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+).$$

(For general coefficients, similar construction)

This gives a functor $\mathrm{Rep}_E(G_{\mathbb{Q}_p}) \to \mathrm{B-Pairs}$, but the latter category is larger !

Definition (Colmez)

A continuous representation V of $G_{\mathbb{Q}_p}$ is trianguline if $W(V)$ is a successive extension of B-pairs of rank 1.

Alternatively : a B-pair is a vector bundle on the Fargues–Fontaine curve. It is triangulable if it is a successive extension of line bundles.

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Overconvergent Fontaine–Mazur conjecture

Theorem (Emerton+...)

A 2-dimensional continuous representation of $G_{\mathbb{Q}}$ is

- *unramified at almost all primes,*
- *triangulline at p,*

(+technical assumptions) if and only if corresponds, up to twist, to a point of a family of eigenforms of finite slope.

In general such a point does not correspond to a "classical" eigenform, but to an "overconvergent" one.

On the other hand, all representations attached to eigenforms are *potentially* trianguline : they become trianguline after restriction to G_E for some finite extension $E/\mathbb{Q}_p$.

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Proof of the main theorem

To prove our main theorem : start with a family $\mathcal{F}$ of p-supercuspidal eigenforms. Let

$$\rho_{\mathcal{F}} \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Z}}_p[[T]])$$

be its associated Galois representation.

Step 1 : $\rho_{\mathcal{F}}|_{G_{\mathbb{Q}_p}}$ is not trianguline, but there exists a quadratic extension E of $\mathbb{Q}_p$ such that $\rho_{\mathcal{F}}|_{G_E}$ is trianguline.

We know this is true for every eigenform in the family. Results of Liu and Kedlaya–Pottharst–Xiao guarantee that triangulations can be interpolated (use the interpolation property of semistable Frobenius eigenvalues).

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Every 2-dimensional, potentially trianguline representation of $G_{\mathbb{Q}_p}$ is of one of the following types :

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2. *a twist of a representation attached to an eigenform,*
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We obtain that $\rho_{\mathcal{F}}|_{G_{\mathbb{Q}_p}}$ is induced by a character

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Via Langlands base change (Arthur–Clozel) we can attach to $\mathcal{F}$ a family $\mathcal{F}_L$ of Hilbert modular forms for $\mathrm{GL}_{2/L}$, with associated representation

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If $\rho_{\mathcal{G}}|_{G_E}$ is the sum of two characters, then $\mathcal{G}$ is a Hilbert CM family.

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Further remarks

- It's possible that the same technique can be adapted to the case where the base field $\mathbb{Q}$ is replaced by a totally real field.
- Coleman–Stein construct a sequence of finite slope eigenforms of growing level that converges to an infinite slope twist of a finite slope eigenform.
- Diao–Liu prove that there cannot be a family of finite slope eigenforms of fixed level converging to an eigenform of infinite slope (also by working on the Galois side).
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Gràcies per la vostra atenció !