From the Generalized Fermat Equation to Hilbert modular forms with prescribed inertial types

(work in progress with L. Dembélé and J. Voight)

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Part (I)

The Generalized Fermat Equation

Motivation

Consider the **Generalized Fermat Equation**

$$Ax^p + By^q = Cz^r, \quad p, q, r \in \mathbb{Z}_{\geq 2}$$

with $A, B, C \in \mathbb{Z}$ non-zero pairwise coprime.

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Conjecture

Fix $A, B, C \in \mathbb{Z}$ pairwise coprime. There exist only finitely many triples (a^p, b^q, c^r) with $(a, b, c) \in (\mathbb{Z} \setminus \{0\})^3$ and p, q, r primes such that:

(i) $1/p + 1/q + 1/r < 1$

(ii) $\gcd(a, b, c) = 1$

(iii) $Aa^p + Bb^q = Cc^r$

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Question: Can one use Hilbert modular forms to solve more cases?

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The main steps of the modular method are:

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- **Contradiction:** Compute all the Hilbert newforms in the predicted spaces. Show that

$$\overline{\rho}_{E,p} \sim \overline{\rho}_{\mathfrak{f},p}$$

does not hold for any of the computed newforms $\mathfrak{f}$.

Example: a result for $r = 13$

Theorem (Dieulefait–F.)

Let $d = 3, 5, 7$ or 11 and γ be an integer divisible only by primes $\ell \not\equiv 1 \pmod{13}$. Set $\mathcal{L} = \{2, 3, 5, 7, 11, 13, 19, 23, 29, 71\}$. Then, the equation

$$x^{13} + y^{13} = d\gamma z^p, \qquad p \notin \mathcal{L}$$

has no non-trivial solutions such that $\gcd(a, b, c) = 1$ and $13 \nmid c$.

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There is some constant M such that if $p > (1 + 3^{18})^2$ and $p \nmid M$ then the equation $x^7 + y^7 = 3z^p$ has no non-trivial solutions (a, b, c) such that $\gcd(a, b) = 1$.

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Question: Can we say something about $x^{19} + y^{19} = Cz^p$?

Answer: For this we need to compute newforms inside a space of cuspforms of dimension above 400000?! With the current state of implementations only to initiate a Hecke operator requires tens or hundreds GB of RAM!

Part (II)

Galois inertial types

Galois Inertial types

Let K be a totally real field. For $\mathfrak{q}$ a prime in K write $I_\mathfrak{q}$ for the inertia group at $\mathfrak{q}$.

Definition

Let $\mathfrak{f}$ be a cuspidal HMF over K of parallel weight 2 and level $\mathcal{N}$. Let $\sigma_{\mathfrak{f},\mathfrak{q}} : W_{K_\mathfrak{q}} \to \mathrm{GL}_2(\mathbb{C})$ be its associated Weil representation at $\mathfrak{q}$. We will say that the restriction

$$\tau_\mathfrak{q} := \sigma_{\mathfrak{f},\mathfrak{q}}|_{I_\mathfrak{q}}$$

is an **inertial type at $\mathfrak{q}$**.

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Problem 1: Given $\mathfrak{f}$ and $\mathfrak{q} \mid \mathcal{N}$ compute the type of $\mathfrak{f}$ at $\mathfrak{q}$?

Problem 2: Given $\mathcal{N}$ and a type $\tau_{\mathfrak{q}}$ at $\mathfrak{q} \mid \mathcal{N}$. Compute all $\mathfrak{f} \in S_2(\mathcal{N})$ with type $\tau_{\mathfrak{q}}$ without computing the whole $S_2(\mathcal{N})$.

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MOST **supercuspidal inertial types** have the following shape

$$\tau_{\mathfrak{q}} := (\mathrm{Ind}_{W_M}^{W_{K_{\mathfrak{q}}}} \chi)|I_{\mathfrak{q}}$$

where $M/K_{\mathfrak{p}}$ is quadratic and χ is a character of W_M such that $\chi \neq \chi^{\sigma}$, where σ is the non-trivial element of $\mathrm{Gal}(M/K_{\mathfrak{p}})$.

Computing types of classical modular forms

The space $S_2(132 = 2^2 \cdot 3 \cdot 11)$ is of dimension 19 and contains two newforms, say f, g.

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Using the L–W implemented in SAGE we compute $\sigma_2(f)$, the automorphic type of f at 2:

```
>new=Newforms(132,names='a');
>pi = LocalComponent(new[0], 2);
>pi.species()
'Supercuspidal'
>pi.characters()
[Character of unramified extension Q_2(s)*(s^2 + s + 1 = 0)
of level 1, mapping s |--> d, 2 |--> 1,
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Then $\tau_2(f) = \mathrm{rec}(\sigma_2)$ corresponds to σ_2 via Local Langlands.

Part (III)

Quaternionic modular forms and Quaternionic types

Quaternionic modular forms

Let B/K be a totally definite quaternion algebra of discriminant $\mathcal{D}$ and $\mathcal{O} \subset \mathcal{O}_0(1) \subset B$ an Eichler order of level $\mathcal{N}$ coprime to $\mathcal{D}$.

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Definition

A **quaternionic modular form for B of parallel weight 2 and level $\mathcal{N}$** is a map

$$f : \mathrm{Cl}(\mathcal{O}) \to \mathbb{C}.$$

We write $M(\mathcal{O})$ for the $\mathbb{C}$-vector space consisting of these forms.

Theorem

For primes $\mathfrak{p} \nmid \mathcal{N}\mathcal{D}$ there are commuting Hecke operator $T_{\mathfrak{p}}$ acting on $M(\mathcal{O})$.

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Theorem (Eichler–Shimizu–Jacquet–Langlands)

Let B/K be a quaternion algebra of discriminant $\mathcal{D}$. Suppose $\mathrm{Cl}^+(K)$ is trivial. Let $\mathcal{O}$ be Eichler order of level $\mathcal{N}$ coprime to $\mathcal{D}$.

Let $S(\mathcal{O}) \subset M(\mathcal{O})$ be the subspace orthogonal to the constant functions. Then, there is an injective map of Hecke modules

$$S(\mathcal{O}) \hookrightarrow \mathcal{S}_2(\mathcal{D}\mathcal{N})$$

whose image consists of Hilbert cuspforms which are **new** at all primes $\mathfrak{p} \mid \mathcal{D}$.

- By this correspondence it is possible to compute HMF by using appropriate quaternion algebras.

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Set $\mathcal{N} = (1)$ thus $\mathcal{O} = \mathcal{O}_0(\mathcal{N}) = \mathcal{O}_0(1)$.

We find that $\#Cl(\mathcal{O}) = 4$, with representatives

$$I_1 = 5\mathcal{O},\ I_2 = 5\mathcal{O} + (2 + i + k)\mathcal{O},$$
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Thus $M(\mathcal{O}) = \mathrm{Map}(\mathrm{Cl}(\mathcal{O}), \mathbb{C}) \cong \mathbb{C}^4$.

Example over $\mathbb{Q}$ (continued)

For each n coprime to $D = 66 = 2 \cdot 3 \cdot 11$, we can write down Hecke operators. For example,

$$T(7) = \begin{pmatrix} 0 & 2 & 2 & 4 \\ 3 & 2 & 0 & 3 \\ 3 & 0 & 2 & 3 \\ 4 & 2 & 2 & 0 \end{pmatrix}$$

has characteristic polynomial $(t-8)(t-2)(t+2)(t+4)$.

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Obervation

Note $\dim S_2(\Gamma_0(66)) = 9$ but the matrix above is 4x4; this is because the JL correspondence allows to work directly on the D-new subspace; this is already the whole $S_2(\Gamma_0(66))^{\text{new}}$.

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Let $B_{\mathfrak{q}}$ be the unique division quaternion algebra over $K_{\mathfrak{q}}$ with maximal order $\mathcal{O}_{\mathfrak{q}}$. A **local quaternionic type at** $\mathfrak{q}$ is an irreducible finite dimensional representation $\rho : \mathcal{O}_{\mathfrak{q}}^* \to \mathsf{GL}(V)$.

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Definition

Let $\mathrm{Cl}(\mathcal{O}) = \{[I_i]\}$ and set $\Gamma_i = \mathcal{O}_L(I_i)^*$. Let ρ be a type at $\mathfrak{q} \mid \mathcal{D}$. A **quaternionic form of weight 2, level $\mathcal{N}$ and type** ρ is a map

$$f \in M(\mathcal{O}, \rho) = \bigoplus_i \mathrm{Map}_{\Gamma_i}(\mathrm{Cl}(\mathcal{O}), V) \cong \bigoplus_i V^{\Gamma_i}.$$

For $\mathfrak{p} \nmid \mathcal{D}\mathcal{N}$ there are Hecke operators $T_{\mathfrak{p}}$ acting on $M(\mathcal{O}, \rho)$.

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- Recall that $B = \left(\frac{-1,-33}{\mathbb{Q}}\right)$ is ramified at 2, so B_2 is a division algebra. Let $J_2 \subset \mathcal{O}_{B_2}$ be the unique maximal two-sided ideal; the quotient is $\mathcal{O}_{B_2}/J_2 \cong \mathbb{F}_4$.
- Choose a cubic character $\chi : \mathbb{F}_4^\times \to \langle \zeta_3 \rangle \subset \mathbb{C}^\times$.

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- $V^{\Gamma_i} = \mathbb{C}$ for $i = 1, 4$ and $V^{\Gamma_i} = \{0\}$ for $i = 2, 3$
- **Thus** $M(\mathcal{O}, \rho)$ **has dimension 2**.

Example over $\mathbb{Q}$ (continued)

- Recall that $B = \left(\frac{-1,-33}{\mathbb{Q}}\right)$ is ramified at 2, so B_2 is a division algebra. Let $J_2 \subset \mathcal{O}_{B_2}$ be the unique maximal two-sided ideal; the quotient is $\mathcal{O}_{B_2}/J_2 \cong \mathbb{F}_4$.
- Choose a cubic character $\chi : \mathbb{F}_4^\times \to \langle \zeta_3 \rangle \subset \mathbb{C}^\times$.
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- **Thus $M(\mathcal{O}, \rho)$ has dimension 2**.
- For example, since $7 \nmid 66$ we obtain the Hecke operator

$$T(7) = \begin{pmatrix} 0 & 2\zeta_3^{-1} \\ -2\zeta_3 & 0 \end{pmatrix}$$

with characteristic polynomial $(t-2)(t+2)$.

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There is a Hecke-equivariant map

$$M(\mathcal{O}, \rho) \xrightarrow{\sim} S_2(\mathfrak{q}^n \mathcal{D}' \mathcal{N}, \tau_\rho) \subset S_2(\mathfrak{q}^n \mathcal{D}' \mathcal{N})^{\mathfrak{q}-\mathrm{new}}.$$

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where $S_2(\mathfrak{q}^n \mathcal{D}' \mathcal{N}, \tau_\rho)$ is generated by the newforms in $S_2(\mathfrak{q}^n \mathcal{D}' \mathcal{N})^{\mathfrak{q}-\mathrm{new}}$ with inertia type τ_ρ.

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$$f(q) = q - q^3 + 2q^5 + 2q^7 + q^9 - q^{11} + 6q^{13} + \dots,$$
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The Main Point

We have worked with matrices of size 2 instead of size 19!!

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We want to take advantage of the following lemma.

Lemma

To a solution of (1) we can attach a Frey curve $E = E_{(a,b)}/K$, where K is the totally real cubic subfield of $\mathbb{Q}(\zeta_{19})$. The prime 2 is inert in K. The Frey curve E is supercuspidal at (2) and has Serre conductor $2^4\pi_{19}^2$.

Happy Birthday!!!

Nuria Vila
Teresa Crespo
Angela Arenas
Enric Nart