Universal quadratic forms and indecomposables over totally real fields

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About us

UFOCLAN: Universal quadratic forms and class numbers

- **Czech Science Foundation** project (2021-2025).
- Principal Investigator: Vítězslav Kala.
- Postdocs: Matteo Bordignon, Giacomo Cherubini, Daniel Gil Muñoz, Eric Nathan Stucky, Pavlo Yatsyna, Błażej Żmija.
- PhD students: Jakub Krásenský, Ester Sgallová, Mikuláš Zindulka.
- Research interests: Universal quadratic forms, indecomposables and their connection to class numbers.

We have a 3 year-postdoc opening (deadline: 15th February).

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Definition

- *Q represents $a \in \mathbb{Z}$ if there is some $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ such that $Q(x_1, \ldots, x_n) = a$.*
- *Q is **universal** if it represents all positive integers.*

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Example (Lagrange, 1770)

The sum of four squares is universal.

It can be checked that

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There are no positive definite ternary universal quadratic forms over $\mathbb{Q}$.

Then, the minimal rank of a positive definite universal quadratic form with integer coefficients is 4.

Theorem (Bhargava - Hanke, 2011)

Let Q be a positive definite quadratic form over $\mathbb{Z}$. If Q represents the twenty nine integers

$$1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26,$$

$$29, 30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, 290,$$

then Q is universal.

Moreover, this set is minimal for that property.

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This is commonly known as the 290-theorem.

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For $\alpha, \beta \in K$, we define

$$\alpha \prec \beta \iff \sigma_i(\alpha) < \sigma_i(\beta), \quad 1 \leq i \leq d.$$

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- *An element $\alpha \in K$ is said to be* ***totally positive*** *if $\alpha \succ 0$.*
- *We say that Q is* ***universal*** *if it represents all totally positive integers.*

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A quadratic form over K is ***positive definite*** *if* $Q(x_1, \ldots, x_n) \succ 0$ *for every* $(x_1, \ldots, x_n) \in \mathcal{O}_K^n$.

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The sum of three squares is universal over $\mathbb{Q}(\sqrt{5})$.

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Theorem (Maaß, 1941)

The sum of three squares is universal over $\mathbb{Q}(\sqrt{5})$.

Theorem (Siegel, 1945)

If the sum of n squares is universal over K, then $K = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{5})$.

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There are not universal forms of ranks 1 or 2, i.e. $m(K) \geq 3$.

Conjecture (Kitaoka)

$m(K) = 3$ only for finitely many totally real fields K.

NUMBER FIELDS WITHOUT n-ARY UNIVERSAL QUADRATIC FORMS

VALENTIN BLOMER AND VÍTĚZSLAV KALA

Abstract. Given any positive integer M, we show that there are infinitely many real quadratic fields that do not admit universal quadratic forms with even cross coefficients in M variables.

UNIVERSAL QUADRATIC FORMS AND ELEMENTS OF SMALL NORM IN REAL QUADRATIC FIELDS

VÍTĚZSLAV KALA

Abstract. For any positive integer M we show that there are infinitely many real quadratic fields that do not admit M-ary universal quadratic forms (without any restriction on the parity of their cross coefficients).

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Proposition

Assume that there are totally positive integers $a_1, \ldots, a_n \in \mathcal{O}_K$ such that if $4a_ia_j \succeq c^2$ for all $1 \leq i, j \leq n$ with $c \in \mathcal{O}_K$, then $c = 0$. Then every universal quadratic form has rank at least n.

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A good choice for these elements a_i are indecomposable elements of K.

Definition

*A totally positive element $\alpha \in \mathcal{O}_K$ is said to be **indecomposable** if $\alpha \neq \beta + \gamma$ for all totally positive $\beta, \gamma \in \mathcal{O}_K$.*

Intuition

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Then the rank of a diagonal universal quadratic form is at least the number of indecomposables modulo squares.

Indecomposables of $K = \mathbb{Q}(\sqrt{D})$ can be determined explicitly in terms of the continued fraction expansion of

$$\omega = \begin{cases} \frac{-1+\sqrt{D}}{2} & \text{if } D \equiv 1 \pmod{4}, \\ \sqrt{D} & \text{if } D \equiv 2,3 \pmod{4}. \end{cases}$$

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For every $i \geq 0$, $\alpha_{i+2} = u_{i+2}\alpha_{i+1} + \alpha_i$.

Theorem

The elements

$$\alpha_{i,r} = r\alpha_{i+1} + \alpha_i,\ 0 \leq r < u_{i+2},\ i \text{ odd}$$

and their conjugates are all indecomposables > 1 *of* $K = \mathbb{Q}(\sqrt{D})$.

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All of this led to the following.

Theorem (Blomer-Kala, 2015; Kala, 2016)

For every $n \in \mathbb{Z}_{\geq 0}$, there are infinitely many totally real quadratic fields K such that $m(K) \geq n$.

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- (**Yatsyna, 2019**) Cubic fields.
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- (**Kala, 2021**) Fields of degree divisible by 2 or 3.

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Theorem (Blomer - Kala, 2018)

Let $K = \mathbb{Q}(\sqrt{D})$. We have

$$\max\left(\frac{M_D}{\kappa s}, C_\epsilon M^*_{D,\epsilon}\right) \leq m_{\text{diag}}(K) \leq 8M_D$$

for any $\epsilon \geq 0$, where:

- *M_D is a sum of the coefficients u_i in the continued fraction expansion of ω.*
- *$\kappa = 2$ if s is odd and $\kappa = 1$ otherwise.*
- *C_ϵ is a constant depending on ϵ.*
- *$M^*_{D,\epsilon}$ is obtained from M_D by removing the summands not satisfying a lower bound involving D and ϵ.*

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UNIVERSAL QUADRATIC FORMS, SMALL NORMS AND TRACES IN FAMILIES OF NUMBER FIELDS

VÍTĚZSLAV KALA AND MAGDALÉNA TINKOVÁ

ABSTRACT. We obtain good estimates on the ranks of universal quadratic forms over Shanks' family of the simplest cubic fields and several other families of totally real number fields. As the main tool we characterize all the indecomposable integers in these fields and the elements of the codifferent of small trace. We also determine the asymptotics of the number of principal ideals of norm less than the square root of the discriminant.

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Theorem (Shintani's unit theorem)

The fundamental domain of the multiplication action by totally positive units of K on the totally positive octant $\mathbb{R}^{d,+}$ is a polyhedric cone.

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On the other hand, an algebraic integer $\alpha \in \mathcal{O}_K$ embeds to $\mathbb{Z}^d$.

Corollary

For each indecomposable $\alpha \in \mathcal{O}_K^+$, there is a totally positive unit $\epsilon \in \mathcal{O}_K^+$ such that $\alpha\epsilon$ lies in a set of the form

$$\bigsqcup_i \mathcal{D}(\alpha_1^{(i)}, \dots, \alpha_e^{(i)}) \cap \mathbb{Z}^d.$$

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Proposition (Thomas, Vasquez (1980))

Let (ϵ_1, ϵ_2) be a proper pair of fundamental units of K. Then, a fundamental domain for the action above is

$$\mathcal{C}(1, \epsilon_1, \epsilon_2) \sqcup \mathcal{C}(1, \epsilon_1, \epsilon_1\epsilon_2^{-1}).$$

Proposition

Let K be a totally real cubic field and let $\{\epsilon_1, \epsilon_2\}$ be a system of fundamental units. For every indecomposable $\alpha \in \mathcal{O}_K$, α lies (up to multiplication by totally positive unit) in

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If K is quadratic, the converse is true.

A simplest cubic field is a cubic number field K given by

$$f(x) = x^3 - ax^2 - (a+3)x - 1,\ a \in \mathbb{Z}_{\geq -1},\ a \neq 0.$$

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This one satisfies

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Theorem (Kala - Tinková, 2020)

Let K be a simplest cubic field such that $\mathcal{O}_K = \mathbb{Z}[\rho]$. Then:

(1) $m_{\mathrm{diag}}(K) \leq 3(a^2 + 3a + 6)$.

(2) $m_{\mathrm{classic}}(K) \geq \frac{a^2+3a+8}{6}$.

(3) *If* $a \geq 21$, $m(K) \geq \frac{\sqrt{a^2+3a+8}}{3\sqrt{2}}$.

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3. *If $a\geq 21$, $m(K)\geq\frac{\sqrt{a^2+3a+8}}{3\sqrt{2}}$.*

In particular, Kitaoka's conjecture holds for this family.

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Let F be a totally real field with Pythagoras number s. Call $\mathcal{S}$ the set of representatives of classes of indecomposables in $\mathcal{O}_F$ modulo $\mathcal{O}_F^{*2}$.
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A simplest cubic field K has $s = 6$ and $|\mathcal{S}| = \frac{a^2+3a+6}{2}$.

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- $a \equiv 3, 21 \pmod{27}$, $\frac{\Delta}{27}$ square-free.
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Next: Codifferent trace, Pythagoras number, bounds for $m(K)$...

- M. Bhargava, J. Hanke; *Universal quadratic forms and the 290-theorem*, Invent. Math., 2005.
- V. Blomer, V. Kala; *Number fields without n-ary universal quadratic forms*, Math. Proc. Cambridge Philos. Soc. **159** (2015), 239-252.
- V. Blomer, V. Kala; *On the rank of universal quadratic forms over real quadratic fields*, Doc. Math. **23** (2018), 15-34.
- V. Kala; *Universal quadratic forms and elements of small norm in real quadratic fields*, Bull. Aust. Math. Soc. **94** (2016), 7-14.
- V. Kala; *Number fields without universal quadratic forms of small rank exist in most degrees*, Preprint.
- V. Kala, J. Svoboda; *Universal quadratic forms over multiquadratic fields*, Ramanujan J. **48** (2019), 151-157.

- V. Kala, M. Tinková; *Universal quadratic forms, small norms and traces in families of number fields*, Preprint.
- H. Maaß; *Über die Darstellung total positiver Zahlen des Körpers $R(\sqrt{5})$ als Summe von drei Quadraten*, Abh. Math. Sem. Univ. Hamburg **14** (1941), 185-191.
- C. L. Siegel; *Sums of m-th powers of algebraic integers*, Annals of Mathematics **46** (1945), 313-339.
- M. Tinková; *Trace and norm of indecomposable integers in cubic orders*, Preprint.
- P. Yatsyna; *A lower bound for the rank of a universal quadratic form with integer coefficients in a totally real field*, Comment. Math. Helvet. **94** (2019), 221-239.

Thank you for your attention