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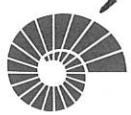
On Selmer Groups and Factoring p -adic L -functions
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Topic : Iwasawa theory + Hida theory

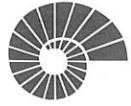


the main conjecture of
Iwasawa theory predicts
a relationship between

↙ p-adic L-functions
(analytic objects)

→ Selmer groups
(algebraic objects)

- In 1980, Gross proved a factorization formula involving Katz's p-adic L-function (a 2-variable p-adic L-function)
- In 1982, Greenberg proved the corresponding result on the algebraic side.
 - ↳ These results provided evidence for the main conjecture for imaginary quadratic fields (before Rubin's proof in 1991)



• In 2014, S. Dasgupta proved a factorization formula involving Rankin - Selberg p-adic L-functions (a 3-variable p-adic L-function)

↳ B. Palvannan's Goal: Prove the corresponding result involving Selmer groups

↳ These results will provide evidence for the main conjectures for the 3-dim'l and 4-dim'l representation.

• \mathbb{Q}_{p^∞} : the cyclotomic \mathbb{Z}_p -extension of \mathbb{Q} .

$$\textcircled{1} \quad (\mathbb{M}_{p^n}) \quad | \quad (\mathbb{Z}/p^n\mathbb{Z})^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}/p^{n-1}\mathbb{Z}$$

(2)

$$\Rightarrow \mathbb{Q}(\mathbb{M}_{p^\infty}) := \bigcup_n \mathbb{Q}(\mathbb{M}_{p^n}) \text{ has}$$

Gal. group \mathbb{Z}_p^\times



$$\text{As } \mathbb{Z}_p^\times \simeq (\mathbb{Z}/p\mathbb{Z})^\times \times (1+p\mathbb{Z}_p) \simeq (\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}_p$$

↳ By the Gal. correspondence $\exists Q_{\infty}$ s.t.

$$\mathcal{O}(M_{p^\infty})$$

Rk: Moreover this is the
unique \mathbb{Z}_p -ext of \mathbb{Q} .

$$\begin{array}{c} \mathbb{Z}_p + \text{Gal}(\mathbb{Q}_{\infty}/\mathbb{Q}) + \text{cts action} \\ \hline \mathbb{Z}_p[\text{Gal}(\mathbb{Q}_{\infty}/\mathbb{Q})] \end{array} \quad \begin{array}{c} \mathbb{Q}_{\infty} \\ \downarrow \\ \mathbb{Q} \end{array} \quad \begin{array}{c} \mathcal{O} \\ \downarrow \\ \mathbb{Z}_p \end{array}$$

$$\mathbb{Z}_p[[\text{Gal}(\mathbb{Q}_{\infty}/\mathbb{Q})]] \simeq \mathbb{Z}_p[[S]] : \begin{array}{l} \text{a complete Noetherian} \\ \text{local ring of Krull dim 2} \\ \text{cyclotomic variable} \end{array}$$

Hida theory

- $p \neq 5$ prime

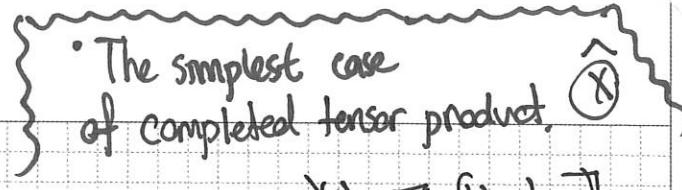
Suppose we have a Gal. repres: - $\rho_F : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{Z}_p[[t]])$

weight variable

↳ Under some hypotheses ρ_F satisfies the Panchishkin condition. (Irr, p -Dis)



The simplest case
of completed tensor product.



• Representations :

- Consider the two maps $\mathbb{Z}_p[[k]] \hookrightarrow \mathbb{Z}_p[[k_1,k_2]]$

$$k \rightarrow k_1 \rightarrow k_2 \hookrightarrow \mathbb{Z}_p[[k_1,k_2]]$$

$$\hookrightarrow \sigma_1, \sigma_2 : G_{\mathbb{Q}} \xrightarrow{\ell_F} GL_2(\mathbb{Z}_p[[k]]) \hookrightarrow GL_2(\mathbb{Z}_p[[k_1,k_2]])$$

\hookrightarrow We get a 4-dim'l Galois repres.

$$G \curvearrowright M_2(A)$$

$$\ell_3 : G_{\mathbb{Q}} \rightarrow GL_4(\mathbb{Z}_p[[k_1,k_2]])$$

$$G \rightarrow \text{Aut}(M_2(A))$$

by the action of $G_{\mathbb{Q}}$ on $M_2(\mathbb{Z}_p[[k_1,k_2]])$.

$$\left\{ \begin{array}{l} \text{choosing---} \\ \text{a basis} \end{array} \right. \begin{array}{l} \text{---} \\ GL_4(A) \end{array}$$

$$G_{\mathbb{Q}} \times M_2(\mathbb{Z}_p[[k_1,k_2]]) \rightarrow M_2(\mathbb{Z}_p[[k_1,k_2]])$$

$$(g, A) \mapsto \sigma_1(g) A \sigma_2(g)^{-1}$$

- For ℓ_3 ,

- $\mathcal{O}_{\ell_3} : 3\text{-variable } p\text{-adic L-function}$
- $\in \text{Frac}(\mathbb{Z}_p[[k_1,k_2]][[s]])$
- two weight variables + one cyclotomic variable

$\text{Sel}_{\ell_3}(\mathbb{Q}_{\infty}) : \text{Selmer group}$

assumed
to be $\sim \text{co. f.g. torsion module over } \mathbb{Z}_p[[k_1,k_2]][[s]]^n$

?



We can think of this map as setting the two weight variables to equal each other.

Consider the map $\pi: \mathbb{Z}_p[[k_1, k_2]] \rightarrow \mathbb{Z}_p[[k]]$
 $k_1, k_2 \mapsto k$

$$\hookrightarrow \pi \circ \ell_3: G_Q \rightarrow GL_4(\mathbb{Z}_p[[k]])$$

gives an action of G_Q on $M_2(\mathbb{Z}_p[[k]])$:

$$G_Q \times M_2(\mathbb{Z}_p[[k]]) \rightarrow M_2(\mathbb{Z}_p[[k]])$$

$$(g, A) \mapsto \ell_F(g) A \ell_F^{-1}(A)$$

After composing with π , the difference between σ_1 and σ_2 :

$$\hookrightarrow \text{Ad}(\ell_F) \stackrel{(*)}{\sim} \text{Ad}^o(\ell_F) \oplus 1$$

$$M_2(\mathbb{Z}_p[[k]])$$

Trace-zero
adjoint

Scalars

$$\sigma_1, \sigma_2 \rightsquigarrow \ell_F$$

$$\Downarrow$$

$$\sigma_1 A \sigma_2^{-1} \rightsquigarrow \ell_F A \ell_F^{-1}$$

$$\ell_2: G_Q \rightarrow GL_3(\mathbb{Z}_p[[k]])$$

a 3-dim'l repres

$$\ell_1: G_Q \rightarrow GL_1(\mathbb{Z}_p[[k]])$$

a 1-dim'l repres

$$\cdot \begin{pmatrix} * & * \\ * & * \end{pmatrix} \simeq \begin{pmatrix} a & * \\ * & -a \end{pmatrix} \oplus \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$$

3 ind. entries

1 entry



- For ℓ_2 ,

$\hookrightarrow \mathcal{O}_{\ell_2}$: 2-variable p-adic L-function

• $\in \text{Frac}(\mathbb{Z}_p[[k]][s])$

• one weight variable + one cyclotomic variable

$\hookrightarrow \text{Sel}_{\ell_2}(\mathbb{Q}_\infty)$: Selmer group

assumed

to be $\simeq \text{co. f.g. torsion module over } \mathbb{Z}_p[[k]][s]$

- For ℓ_1 ,

$\hookrightarrow \mathcal{O}_{\ell_1}$: 1-variable p-adic L-function

• $\in \text{Frac}(\mathbb{Z}_p[[k]][s])$ (Actually, it lives in a smaller ring)

• one cyclotomic variable + constant in the weight variable

$\hookrightarrow \text{Sel}_{\ell_1}(\mathbb{Q}_\infty)$: Selmer groups

Due to a result of Iwasawa $\simeq \text{co. f.g. torsion modules over } \mathbb{Z}_p[[k]][s]$

• $\Pi : \mathbb{Z}_p[[k_1 k_2]][s] \rightarrow \mathbb{Z}_p[[k]][s]$

$k_1 k_2 \mapsto k$

$\hookrightarrow \overline{\Pi} \circ \ell_3 \simeq \ell_2 \oplus \ell_1$ by (*)



"Div" is a substitute for the notion of characteristic ideal.

Main Conjectures

- $\text{Div}(-) \in \text{free abel. gp on the prime ideals of height } 1$.
- $f \in \text{Fract} \Rightarrow \text{div}(f) := \text{div}(-/f-)$

- MC- ℓ_1 : Thm (Mazur-Wiles, 1984):

In the divisor group, $\text{Div}(\mathcal{O}_{\ell_1}) + E_{\ell_1} = \text{Div}(\text{Sel}_{\ell_1}(\mathbb{Q}_\infty))$
of $\mathbb{Z}_p[[t]][[s]]$

- MC- ℓ_2 : Main conj. for ℓ_2

In the divisor group, $\text{Div}(\mathcal{O}_{\ell_2}) + E_{\ell_2} \stackrel{?}{=} \text{Div}(\text{Sel}_{\ell_2}(\mathbb{Q}_\infty))$
of $\mathbb{Z}_p[[t]][[s]]$

- MC- ℓ_3 : Main conj. for ℓ_3

In the divisor group, $\text{Div}(\mathcal{O}_{\ell_3}) + E_{\ell_3} \stackrel{?}{=} \text{Div}(\text{Sel}_{\ell_3}(\mathbb{Q}_\infty))$
of $\mathbb{Z}_p[[t_1 t_2]][[s]]$

- E_{ℓ_1} : Error terms that depend on local factors away from p and poles of the p -adic L -function.

- \nexists main conjecture for $\pi \circ \rho_3$ as it doesn't satisfy the "Panchiskin condition".

- p -adic L -functions: primitive

Selmer groups: non-primitive (it is quite easier to establish that they satisfy better alg. prop.)

Main theorems:

Thm (Dasgupta, 2014): In $\text{Frac}(\mathbb{Z}_p[[k]][s])$,

$$\pi(\Theta_{\ell_3}) = \Theta_{\ell_2} \cdot \Theta_{\ell_1}$$

Thm (Palvannan, 2016): In the divisor group of $\mathbb{Z}_p[[k]][s]$,

$$\text{Div}(\text{Sel}_{\pi \circ \ell_3}(\mathbb{Q}_{\infty})) = \text{Div}(\text{Sel}_{\ell_2}(\mathbb{Q}_{\infty})) + \text{Div}(\text{Sel}_{\ell_1}(\mathbb{Q}_{\infty})).$$

Q: What is the relationship between $\pi(\Theta_{\ell_3})$ and $\text{Sel}_{\pi \circ \ell_3}(\mathbb{Q}_{\infty})$?

Thm (Palvannan, 2016): Suppose we have the following

(in) equality of divisors in $\mathbb{Z}_p[[k_1, k_2]][s]$.

$$\text{Div}(\Theta_{\ell_3}) + E_{\ell_3} \stackrel{(\geq)}{\equiv} \text{Div}(\text{Sel}_{\ell_3}(\mathbb{Q}_{\infty})) \quad (\text{ES})$$

Then we have the following (in) equality of divisors in $\mathbb{Z}_p[[k_1, k_2]][s]$

$$\text{Div}(\pi(\Theta_{\ell_3})) + E_{\pi \circ \ell_3} \stackrel{(\geq)}{\equiv} \text{Div}(\text{Sel}_{\pi \circ \ell_3}(\mathbb{Q}_{\infty})) \quad (+)$$

Rk: (ES) is expected due to recent work on Euler systems

by Loeffler-Zerbes (2014) \rightarrow Their thm's do not exactly apply in Palvannan's case: one of their hyp. is that the tensor prod. of residual repre's is trivial.

Someone might relax those technical hypot.

but Palvannan's rep. breaks down: L^{+3} dim'

- Thm 1 + Thm 2 + Thm 3 + MC- ℓ_1 implies that

$$\text{Div}(\mathcal{O}_{\ell_2}) + E_{\ell_2} \stackrel{(\star)}{=} \text{Div}(\text{Sel}_{\ell_2}(\mathbb{Q}_\infty)^\vee)$$

in the divisor group of $\mathbb{Z}_p[[t]][[s]]$.

- But also,

Thm (Urban, 2001): Under certain additional hypotheses,

• Selmer groups and Eisenstein-Klingen ideal

$$\text{Div}(\mathcal{O}_{\ell_2}) + E_{\ell_2} \leq \text{Div}(\text{Sel}_{\ell_2}(\mathbb{Q}_\infty)^\vee) \quad (\text{UR})$$

- Under (ES) and (UR), we have $\text{MC-}\ell_2$.

- One more hypothesis $\Rightarrow (= \text{in (ES)} \Leftrightarrow = \text{in (+)})$

$$\hookrightarrow \text{Th3} + \text{Thm 1+2+MC-}\ell_1+\text{MC-}\ell_2 \Rightarrow \text{MC-}\ell_3.$$

hypothesis



Gracias!