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On Selmer Groups and Factoring p-adic L-functions

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Topic : Iwasawa theory + Hida theory

↓

the main conjecture of Iwasawa theory predicts a relationship between

p-adic L-functions (analytic objects)

Selmer groups (algebraic objects)

- In 1980, Gross proved a factorization formula involving Katz's p-adic L-function (a 2-variable p-adic L-function)
- In 1982, Greenberg proved the corresponding result on the algebraic side.

↳ These results provided evidence for the main conjecture for imaginary quadratic fields (before Rubin's proof in 1991)

· In 2014, S. Dasgupta proved a factorisation formula involving Rankin-Selberg p-adic L-functions (a 3-variable p-adic L-function)

↳ B. Palvannan's Goal: Prove the corresponding result involving Selmer groups

↳ These results will provide evidence for the main conjectures for the 3-dim'l and 4-dim'l representation.

· $\mathbb{Q}_\infty$: the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.

$$\begin{array}{c} \mathbb{Q}(\mu_{p^n}) \\ | \\ \mathbb{Q} \end{array} \Big) \; \left(\mathbb{Z}/p^n\mathbb{Z}\right)^\times \simeq \left(\mathbb{Z}/p\mathbb{Z}\right)^\times \times \mathbb{Z}/p^{n-1}\mathbb{Z}$$

$\Rightarrow \mathbb{Q}(\mu_{p^\infty}) := \bigcup_n \mathbb{Q}(\mu_{p^n})$ has Gal. group $\mathbb{Z}_p^\times$

As $\mathbb{Z}_p^\times \simeq (\mathbb{Z}/p\mathbb{Z})^\times \times (1+p\mathbb{Z}_p) \simeq (\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}_p$

↳ By the Gal. correspondence $\exists\, \mathbb{Q}_\infty$ s.t.

$$\begin{array}{c} \mathbb{Q}(\mu_{p^\infty}) \\ | \\ \mathbb{Q}_\infty \\ | \\ \mathbb{Q} \end{array} \Big) \mathbb{Z}_p$$

Rk: Moreover this is the unique $\mathbb{Z}_p$-ext of $\mathbb{Q}$.

· $\underbrace{\mathbb{Z}_p + \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) + \text{cts action}}_{\mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]}$

· $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]] \simeq \mathbb{Z}_p[[s]]$: a complete Noetherian local ring of Krull dim 2

s → cyclotomic variable

· Hida theory

- $p \geq 5$ prime

Suppose

- F : Hida Family (p-adic family of cuspidal eigenforms)

we have a Gal. repres :

- $\rho_F : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Z}_p[[t]])$

t → weight variable

↳ Under some hypotheses ρ_F satisfies the Panchishkin condition. (Irr, p-Dis)

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- The simplest case of completed tensor product $\hat{\otimes}$

- Representations:
  - Consider the two maps $\mathbb{Z}_p[[k]] \xrightarrow{k \mapsto k_1} \mathbb{Z}_p[[k_1,k_2]]$, $\mathbb{Z}_p[[k]] \xrightarrow{k \mapsto k_2} \mathbb{Z}_p[[k_1,k_2]]$

    ↳ $\sigma_1, \sigma_2 : G_{\mathbb{Q}} \xrightarrow{\rho_F} GL_2(\mathbb{Z}_p[[k]]) \hookrightarrow GL_2(\mathbb{Z}_p[[k_1,k_2]])$

    ↳ We get a 4-dim'l Galois repres.

$$\rho_3 : G_{\mathbb{Q}} \to GL_4(\mathbb{Z}_p[[k_1,k_2]])$$

- $G \curvearrowright M_2(A)$
  $\}$
  $G \to Aut(M_2(A))$
  choosing a basis $\cong GL_4(A)$

by the action of $G_{\mathbb{Q}}$ on $M_2(\mathbb{Z}_p[[k_1,k_2]])$:

$$G_{\mathbb{Q}} \times M_2(\mathbb{Z}_p[[k_1,k_2]]) \to M_2(\mathbb{Z}_p[[k_1,k_2]])$$

$$(g, A) \mapsto \sigma_1(g)\, A\, \sigma_2(g)^{-1}$$

- For ρ_3,
  - → Θ_{ρ_3} : 3-variable p-adic L-function
    - $\in \mathrm{Frac}(\mathbb{Z}_p[[k_1,k_2]][[s]])$
    - two weight variables + one cyclotomic variable
  - → $\mathrm{Sel}_{\rho_3}(\mathbb{Q}_\infty)$: Selmer group
    - assumed to be ← co.f.g. torsion module over $\mathbb{Z}_p[[k_1,k_2]][[s]]$?

- We can think of this map as setting the two weight variables to equal each other.

- Consider the map $\pi : \mathbb{Z}_p[[k_1, k_2]] \longrightarrow \mathbb{Z}_p[[k]]$

$$k_1, k_2 \longmapsto k$$

↳ $\pi \circ \rho_3 : G_{\mathbb{Q}} \to GL_4(\mathbb{Z}_p[[k]])$

gives an action of $G_{\mathbb{Q}}$ on $M_2(\mathbb{Z}_p[[k]])$:

$$G_{\mathbb{Q}} \times M_2(\mathbb{Z}_p[[k]]) \longrightarrow M_2(\mathbb{Z}_p[[k]])$$

$$(g, A) \longmapsto \rho_F(g)\, A\, \rho_F(A)^{-1}$$

- After composing with π, $\nexists$ difference between σ_1 and σ_2:

$$\sigma_1, \sigma_2 \rightsquigarrow \rho_F$$
$$\Downarrow$$
$$\sigma_1 A \sigma_2^{-1} \rightsquigarrow \rho_F A \rho_F^{-1}$$

$$\underbrace{Ad(\rho_F)}_{M_2(\mathbb{Z}_p[[k]])} \overset{(*)}{\simeq} \underbrace{Ad^{\circ}(\rho_F)}_{\text{Trace-zero adjoint}} \oplus \underbrace{\mathbb{1}}_{\text{scalars}}$$

$\rho_2 : G_{\mathbb{Q}} \to GL_3(\mathbb{Z}_p[[k]])$

a $\underline{3}$-dim'l repres

$\rho_1 : G_{\mathbb{Q}} \to GL_1(\mathbb{Z}_p[[k]])$

a $\underline{1}$-dim'l repres

- $$\begin{pmatrix} * & * \\ * & * \end{pmatrix} \simeq \underbrace{\begin{pmatrix} a & * \\ * & -a \end{pmatrix}}_{\text{3 ind. entries}} \oplus \underbrace{\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}}_{\text{1 entry}}$$

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- For ℓ_2,
  - $\to \Theta_{\ell_2}$: 2-variable p-adic L-function
    - $\in \mathrm{Frac}(\mathbb{Z}_p[[k]][[s]])$
    - one weight variable + one cyclotomic variable
  - $\to \mathrm{Sel}_{\ell_2}(\mathbb{Q}_\infty)$: Selmer group

assumed to be $\leftarrow$ co. f.g. torsion module over $\mathbb{Z}_p[[k]][[s]]$

✓ ?

- For ℓ_1,
  - $\to \Theta_{\ell_1}$: 1-variable p-adic L-function
    - $\in \mathrm{Frac}(\mathbb{Z}_p[[k]][[s]])$ (Actually, it lives in a smaller ring)
    - one cyclotomic variable + constant in the weight variable
  - $\to \mathrm{Sel}_{\ell_1}(\mathbb{Q}_\infty)$: Selmer groups

Due to a result of Iwasawa $\leftarrow$ co. f.g. torsion modules over $\mathbb{Z}_p[[k]][[s]]$

- $\Pi : \mathbb{Z}_p[[k_1,k_2]][[s]] \to \mathbb{Z}_p[[k]][[s]]$

  $k_1, k_2 \mapsto k$

  $\to \Pi \circ \ell_3 \simeq \ell_2 \oplus \ell_1$ by (*)

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• "Div" is a substitute for the notion of characteristic ideal.

• $\mathrm{Div}(-) \in$ free abel. gp on the prime ideals of height 1.

• $f \in \mathrm{Frac}(-) \Rightarrow \mathrm{div}(f) := \mathrm{div}(-/f-)$

Main Conjectures

• MC-ℓ_1 : Thm (Mazur-Wiles, 1984):

In the divisor group of $\mathbb{Z}_p[[t]][[s]]$, $\mathrm{Div}(\Theta_{\ell_1}) + E_{\ell_1} = \mathrm{Div}(\mathrm{Sel}_{\ell_1}(\mathbb{Q}_\infty)^\vee)$

• to a f.g. torsion module over an int. closed Noeth. local domain or a nonzero elem. of it's frac field, we shall associate an elem. of the div. group.

• MC-ℓ_2 : Main conj. for ℓ_2

In the divisor group of $\mathbb{Z}_p[[t]][[s]]$, $\mathrm{Div}(\Theta_{\ell_2}) \neq E_{\ell_2} \overset{?}{=} \mathrm{Div}(\mathrm{Sel}_{\ell_2}(\mathbb{Q}_\infty)^\vee)$

• MC-ℓ_3 : Main conj. for ℓ_3

In the divisor group of $\mathbb{Z}_p[[k_1,k_2]][[s]]$, $\mathrm{Div}(\Theta_{\ell_3}) + E_{\ell_3} \overset{?}{=} \mathrm{Div}(\mathrm{Sel}_{\ell_3}(\mathbb{Q}_\infty)^\vee)$

• $E_{\ell_?}$: Error terms that depend on local factors away from p and poles of the p-adic L-function.

• $\nexists$ main conjecture for $\Pi \circ \ell_3$ as it doesn't satisfy the "Panchishkin condition".

• p-adic L-functions : primitive

Selmer groups : non-primitive (it is quite easier to establish that they satisfy better alg. prop.)

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Main theorems:

Thm (Dasgupta, 2014): In $\mathrm{Frac}(\mathbb{Z}_p[[k]][[s]])$,

$$\pi(\Theta_{\ell_3}) = \Theta_{\ell_2} \cdot \Theta_{\ell_1}$$

Thm (Palvannan, 2016): In the divisor group of $\mathbb{Z}_p[[k]][[s]]$,

$$\mathrm{Div}(\mathrm{Sel}_{\pi\circ\ell_3}(\mathbb{Q}_\infty)^\vee) = \mathrm{Div}(\mathrm{Sel}_{\ell_2}(\mathbb{Q}_\infty)^\vee) + \mathrm{Div}(\mathrm{Sel}_{\ell_1}(\mathbb{Q}_\infty)^\vee).$$

Q: What is the relationship between $\pi(\Theta_{\ell_3})$ and $\mathrm{Sel}_{\pi\circ\ell_3}(\mathbb{Q}_\infty)$?

Thm (Palvannan, 2016): Suppose we have the following (in)equality of divisors in $\mathbb{Z}_p[[k_1,k_2]][[s]]$.

$$\mathrm{Div}(\Theta_{\ell_3}) + E_{\ell_3} \overset{(\geq)}{=} \mathrm{Div}(\mathrm{Sel}_{\ell_3}(\mathbb{Q}_\infty)^\vee) \qquad (ES)$$

Then we have the following (in)equality of divisors in $\mathbb{Z}_p[[k_1,k_2]][[s]]$

$$\mathrm{Div}(\pi(\Theta_{\ell_3})) + E_{\pi\circ\ell_3} \overset{(\geq)}{=} \mathrm{Div}(\mathrm{Sel}_{\pi\circ\ell_3}(\mathbb{Q}_\infty)^\vee) \qquad (+)$$

Rk: (ES) is expected due to recent work on Euler systems by Lei-Loeffler-Zerbes (2014)

→ Their thm's do <u>not</u> exactly apply in Palvannan's case: one of their hyp. is that the tensor prod. of residual repre.'s is irred., but Palvan's rep. breaks down: 1+3 dim'l

Someone might relax ← those technical hypot.

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- Thm 1 + Thm 2 + Thm 3 + MC-ℓ_1 implies that

$$\mathrm{Div}(\Theta_{\ell_2}) + E_{\ell_2} \overset{(?)}{=} \mathrm{Div}(\mathrm{Sel}_{\ell_2}(\mathbb{Q}_\infty)^\vee)$$

in the divisor group of $\mathbb{Z}_p[[t]][[s]]$.

- But also,

Thm (Urban, 2001): Under certain additional hypotheses,

• Selmer groups and Eisenstein-Klingen ideal

$$\mathrm{Div}(\Theta_{\ell_2}) + E_{\ell_2} \leq \mathrm{Div}(\mathrm{Sel}_{\ell_2}(\mathbb{Q}_\infty)^\vee) \qquad (UR)$$

- Under (ES) and (UR), we have MC-ℓ_2.

- One more hypothesis $\Rightarrow$ (= in (ES) $\Leftrightarrow$ = in (+))

↳ This hypothesis + Thm 1 + Thm 2 + MC-ℓ_1 + MC-ℓ_2 $\Rightarrow$ MC-ℓ_3.

Gracias!