Galois representations associated to ordinary Hilbert modular forms: Wiles' theorem

Francesc Fité (Universität Duisburg-Essen)

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Layout

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Notations

- F a totally real field; $\mathcal{O}_F$ its ring of integers; $d := [F : \mathbb{Q}]$.
- $\mathfrak{n} \subseteq \mathcal{O}_F$ integral ideal; $\psi_0 \colon (\mathcal{O}_F/\mathfrak{n})^* \to \overline{\mathbb{Q}}^*$; $\mathfrak{a}$ a fractional ideal.
- $S_k(\Gamma(\mathfrak{a}, \mathfrak{n}), \psi_0) =$ space of Hilbert cusp forms $f \colon \mathcal{H}^d \to \mathbb{C}$ of parallel weight $k \geqslant 1$, character ψ_0, and relative to $\Gamma(\mathfrak{a}, \mathfrak{n})$.
- $h =$ strict class number of F ($\mathfrak{t}_\gamma$ reps. of the strict class ideals).
- $S_k(\mathfrak{n}, \psi_0)$ denotes the space of

$$\mathbf{f} := (f_1, \dots, f_h) \in \prod_{\gamma=1}^{h} S_k(\Gamma(\mathfrak{t}_\gamma \mathfrak{d}, \mathfrak{n}), \psi_0)\,.$$

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Dirichlet series

- Each f_γ admits a Fourier expansion

$$f_\gamma(z_1,\dots,z_d) = \sum_{0\ll\mu\in\mathfrak{t}_\gamma} a_\gamma(\mu) e^{2\pi i(\sum_{j=1}^d \mu_j z_j)}, \quad \text{for } (z_1,\dots,z_d)\in\mathcal{H}^d.$$

($\mu_1,\dots,\mu_d$ denote the images of μ by the d embeddings of F into $\mathbb{C}$).

- For $0 \neq \mathfrak{a} \subseteq \mathcal{O}_F$, there exist $\gamma \in \{1,\dots,h\}$ and a totally positive $\mu \in \mathfrak{t}_\gamma$ such that $\mathfrak{a} = \mu\mathfrak{t}_\gamma^{-1}$. Define

$$c(\mathfrak{a},\mathbf{f}) := a_\gamma(\mu) N(\mathfrak{t}_\gamma)^{-k/2}.$$

(it depends neither on the choice of γ nor of $\mathfrak{t}_\gamma$).

- The *Dirichlet series* associated to $\mathbf{f}$ is

$$D(\mathbf{f},s) := \sum_{0\neq\mathfrak{a}\subseteq\mathcal{O}_F} c(\mathfrak{a},\mathbf{f}) N(\mathfrak{a})^{-s}.$$

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Ordinary primes

- There is a theory of Hecke operators $\{T_{\mathfrak{n}}(\mathfrak{a}), S_{\mathfrak{n}}(\mathfrak{a})\}_{\mathfrak{a}\subseteq\mathcal{O}_F}$ on $S_k(\mathfrak{n},\psi_0)$.
- $\psi_0 \rightsquigarrow \psi\colon I_{\mathfrak{n}\infty}\to\overline{\mathbb{Q}}^*$ ray class char. of modulus $\mathfrak{n}\infty$ restricting to ψ_0 on $(\mathcal{O}_F/\mathfrak{n})^*$. Set

$$S_k(\mathfrak{n},\psi) := \{\mathbf{f}\in S_k(\mathfrak{n},\psi_0)\,|\, S_{\mathfrak{n}}(\mathfrak{a})(\mathbf{f}) = \psi(\mathfrak{a})\mathbf{f} \text{ for all } \mathfrak{a}\subseteq\mathcal{O}_F\}\,.$$

- For $f\in S_k(\mathfrak{n},\psi)$ a newform, set

$$K_{\mathbf{f}} := \mathbb{Q}(\{c(\mathfrak{a},\mathbf{f})\}_{\mathfrak{a}\subseteq\mathcal{O}_F}); \qquad \mathcal{O}_{\mathbf{f}} \text{ its ring of integers.}$$

- $\lambda\subseteq\mathcal{O}_{\mathbf{f}}$ prime. $\mathbf{f}$ is *ordinary at* λ if for $\mathfrak{p}\subseteq\mathcal{O}_F$, $\mathfrak{p}|N(\lambda)$,

$$x^2 - c(\mathfrak{p},\mathbf{f})x + \psi(\mathfrak{p})N(\mathfrak{p})^{k-1}$$

has a unit root mod λ.

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Wiles' result

Main Theorem (Wiles)

If $\mathbf{f}$ is ordinary at λ, there exists a continuous odd irreducible rep.

$$\varrho_{\mathbf{f},\lambda}\colon G_F \to \mathrm{GL}_2(\mathcal{O}_{\mathbf{f},\lambda})$$

unramified outside $\mathfrak{n}N(\lambda)$ and such that for all primes $\mathfrak{q} \nmid \mathfrak{n}N(\lambda)$

$$\mathrm{Tr}(\varrho_{\mathbf{f},\lambda})(\mathrm{Frob}_{\mathfrak{q}}) = c(\mathfrak{q}, \mathbf{f}),$$
$$\det(\varrho_{\mathbf{f},\lambda})(\mathrm{Frob}_{\mathfrak{q}}) = \psi(\mathfrak{q})N(\mathfrak{q})^{k-1}.$$

- In the previous talk we saw:

Theorem (Carayol)

For $k \geqslant 2$ and $\mathbf{f}$ not necessarily ordinary at λ, there exists $\varrho_{\mathbf{f},\lambda}$ if either

i) *d is odd; or*

ii) *d is even and there is a prime $\mathfrak{p}||\mathfrak{n}$.*

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General strategy

- $S_k(\mathfrak{n}, \psi \mid \mathbb{Z}[\psi]) := \{\mathbf{g} \in S_k(\mathfrak{n}, \psi) \mid c(\mathfrak{a}, \mathbf{g}) \in \mathbb{Z}[\psi] \text{ for all } \mathfrak{a} \subseteq \mathcal{O}_F\}$.
- Fix a prime p from now on. For simplicity assume $p \geqslant 3$.
- For a subring $\mathbb{Z}[\psi] \subseteq A \subseteq \overline{\mathbb{Q}}_p$, set

$$S_k(\mathfrak{n}, \psi \mid A) := S_k(\mathfrak{n}, \psi \mid \mathbb{Z}[\psi]) \otimes_{\mathbb{Z}[\psi]} A$$

- $\mathcal{K}$ a finite extension of $\mathbb{Q}_p((X))$; $\Lambda =$ integral closure of $\mathbb{Z}_p[[X]]$. Assume $\Lambda \supseteq \mathbb{Z}_p[\psi][[X]]$.
- $K := \mathcal{K} \cap \overline{\mathbb{Q}}_p$; $\mathcal{O}$ its ring of integers.

Step 1: Λ-adic forms

Define $\mathcal{S}(\overline{\mathfrak{n}}, \psi \mid \Lambda)$ and specialization maps $\nu_{k,r} : \Lambda \to \overline{\mathbb{Q}}_p$ such that

$$\mathcal{F} \in \mathcal{S}(\overline{\mathfrak{n}}, \psi \mid \Lambda) \Rightarrow \nu_{k,r}(\mathcal{F}) \in S_k(\mathfrak{n}p^r, \psi \varrho_{\zeta_{p^r}} \omega^{2-k} \mid \mathcal{O}[\zeta_{p^r}]) .$$

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Step 2: Lifting

Define $S_k^{\mathrm{ord}}(\mathfrak{n},\psi\,|\,\mathcal{O}) \subseteq S_k(\mathfrak{n},\psi\,|\,\mathcal{O})$ (containing the **f** ordinary at $\lambda|p$) s.t.

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Step 3: Patching – Theory of pseudo-representations

Write $\mathbf{f}_{k,r} := \nu_{k,r}(\mathcal{F})$.

There exists $\varrho_{\mathbf{f}_{k,r},\lambda}$ for *infinitely many* $\nu_{k,r}$ $\Longrightarrow$ There exists $\varrho_{\mathcal{F}} : G_F \to \mathrm{GL}_2(\mathcal{K})$ s.t. $\nu_{k,r}(\varrho_{\mathcal{F}}) = \varrho_{\mathbf{f}_{k,r},\lambda}$ for *almost every* $\nu_{k,\zeta}$

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Step 1: Defining Λ-adic forms

- Recall that $p \geqslant 3$ and further assume $[\mathbb{Q}_\infty \cap F : \mathbb{Q}] = 1$.
- For $k \geqslant 1$, $r \geqslant 0$, define the specialization map
$$\nu_{k,r} \colon \mathbb{Z}_p[[X]] \to \mathbb{Z}_p[\zeta]\,, \qquad X \mapsto \zeta(1+p)^{k-2} - 1\,,$$
where $\zeta^{p^r} = 1$.
- By the going-up theorem:
$$\begin{array}{ccccc} P_{k,r} & \subseteq & \Lambda & \subseteq & \mathcal{K} \\ | & & | & & | \\ \ker(\nu_{k,r}) & \subseteq & \mathbb{Z}_p[[X]] & \subseteq & \mathbb{Q}_p((X)) \end{array}$$
- Recall that $K = \mathcal{K} \cap \overline{\mathbb{Q}}_p$ and $\mathcal{O} = \Lambda \cap \overline{\mathbb{Q}}_p$.
- We may view $P_{k,r}$ as the kernel of an $\mathcal{O}$-algebra homomorphism $\nu_{k,r} \colon \Lambda \to \overline{\mathbb{Q}}_p$ extending $\nu_{k,r}$.

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- Recall that $K = \mathcal{K} \cap \overline{\mathbb{Q}}_p$ and $\mathcal{O} = \Lambda \cap \overline{\mathbb{Q}}_p$.
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Step 1: Defining Λ-adic forms

- Recall that $p \geqslant 3$ and further assume $[\mathbb{Q}_\infty \cap F \colon \mathbb{Q}] = 1$.
- For $k \geqslant 1$, $r \geqslant 0$, define the specialization map

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Λ-adic forms

- For a fractional ideal $\mathfrak{a}$ of F s.t. $(\mathfrak{a}, p) = 1$, we can write

$$N(\mathfrak{a}) = (1+p)^{\alpha}\delta\,, \qquad \text{with } \delta \in \mu_{p-1},\ \alpha \in \mathbb{Z}_p\,.$$

- Given $\psi\colon I_{\mathfrak{n}\infty} \to \overline{\mathbb{Q}}^*$ and $\zeta^{p^r} = 1$, define

$$\boldsymbol{\psi}\colon \varprojlim_t I_{\mathfrak{n}p^t} \to \Lambda\,, \quad \boldsymbol{\psi}(\mathfrak{a}) = \psi(\mathfrak{a})(1+X)^{\alpha}\,,$$
$$\varrho_\zeta\colon I_{p^r\mathcal{O}_F} \to \overline{\mathbb{Q}}^*\,, \quad \varrho_\zeta(\mathfrak{a}) = \zeta^{\alpha}\,,$$
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Definition

A Λ-*adic cuspidal form* $\mathcal{F}$ over F of level $\mathfrak{n}$ and character $\psi\colon \varprojlim_t I_{\mathfrak{n}p^t} \to \Lambda$ is a collection of elements of Λ

$$\{c(\mathfrak{a},\mathcal{F})(X)\}_{0\neq\mathfrak{a}\subseteq\mathcal{O}_F} \subseteq \Lambda\,,$$

such that, for almost every $\nu_{k,r}$, with $k \geqslant 2$, $r \geqslant 0$, there exists

$$\mathbf{f}_{\nu_{k,r}} \in S_k(\mathfrak{n}p^r, \psi\varrho_\zeta\omega^{2-k} \,|\, \mathcal{O}[\zeta])$$

whose associated Dirichlet series is

$$D(\mathbf{f}_{\nu_{k,r}}, s) = \sum_{0\neq\mathfrak{a}\subseteq\mathcal{O}_F} \nu_{k,r}(c(\mathfrak{a},\mathcal{F})(X))N(\mathfrak{a})^{-s}.$$

- By abuse of notation, write $\nu_{k,r}(\mathcal{F}) := \mathbf{f}_{\nu_{k,r}}$.
- $\mathcal{S}(\mathfrak{n},\psi\,|\,\Lambda)$ is the Λ-module of Λ-adic cusp forms.
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Step 2: The space of classical p-stabilized forms

- We work at level $\mathfrak{n}p^r$, with $r \geqslant 1$.
- The *Hida operator*

$$e := \lim_{n\to\infty} T_{\mathfrak{n}p^r}(p)^{n!} \colon S_k(\mathfrak{n}p^r, \psi \,|\, \mathcal{O}) \to S_k(\mathfrak{n}p^r, \psi \,|\, \mathcal{O})\,.$$

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- $\mathfrak{P}$:= product of primes of $\mathcal{O}_F$ above p not dividing $\mathfrak{m}$.
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Example

- $F = \mathbb{Q}$ and $N \geqslant 1$ with $(N, p) = 1$.
- Let $f = \sum_{n\geqslant 1} c_n q^n \in S_k(\Gamma_0(N), \psi)$ be an ordinary eigenform.
- $x^2 - c_p(f)x + \psi(p)p^{k-1}$ has roots α, β and suppose that α is a unit.
- Using usual relations between Hecke operators one can check that $T_{Np}(p)$ acts on $\langle f, f(pz)\rangle$ by means of

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Example

- $F = \mathbb{Q}$ and $N \geqslant 1$ with $(N, p) = 1$.
- Let $f = \sum_{n\geqslant 1} c_n q^n \in S_k(\Gamma_0(N), \psi)$ be an ordinary eigenform.
- $x^2 - c_p(f)x + \psi(p)p^{k-1}$ has roots α, β and suppose that α is a unit.
- Using usual relations between Hecke operators one can check that $T_{Np}(p)$ acts on $\langle f, f(pz)\rangle$ by means of

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The Lifting Theorem

- **The space of p-stabilized Λ-adic forms**: There is an idempotent

$$\mathcal{E}\colon\ \mathcal{S}(\overline{\mathfrak{n}},\psi\,|\,\Lambda)\to\mathcal{S}(\overline{\mathfrak{n}},\psi\,|\,\Lambda)$$

of $\mathrm{End}_\Lambda(\mathcal{S}(\overline{\mathfrak{n}},\psi\,|\,\Lambda))$ s.t. for almost every ν we have

$$\nu(\mathcal{E}(\mathcal{F}))=e(\nu(\mathcal{F}))\,.$$

Theorem (Wiles, helped by Taylor & Shimura)

$\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}},\psi\,|\,\Lambda):=\mathcal{E}\,\mathcal{S}(\overline{\mathfrak{n}},\psi\,|\,\Lambda)$ is free Λ-module of finite rank.

- **Hecke operators**: For every $\mathfrak{a}\subseteq\mathcal{O}_F$, one can define Λ-linear maps

$$\mathcal{T}(\mathfrak{a})\colon\ \mathcal{S}(\overline{\mathfrak{n}},\psi\,|\,\Lambda)\to\mathcal{S}(\overline{\mathfrak{n}},\psi\,|\,\Lambda)$$

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- $\mathcal{F} \in \mathcal{S}(\overline{\mathfrak{n}}, \psi \,|\, \Lambda)$ is called:
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Theorem (Hida for $F = \mathbb{Q}$ and $k \geqslant 2$; Wiles in general)

For a newform $\mathbf{f} \in S_k^{\mathrm{ord}}(\mathfrak{n}, \psi\omega^{2-k} \,|\, \mathcal{O})$, $k \geqslant 1$, there exist $\Lambda \supseteq \mathbb{Z}_p[[X]]$, $\nu_{k,0}$, and a newform $\mathcal{F} \in \mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}, \psi \,|\, \Lambda)$ s.t. $\nu_{k,0}(\mathcal{F}) = \mathbf{f}$.

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Step 3: Patching

- Recall the setting

$$\begin{array}{ccc} \Lambda & \subseteq & \mathcal{K} \\ | & & | \\ \mathbb{Z}_p[\psi][[X]] & \subseteq & \mathbb{Q}_p[\psi]((X)) \end{array} \qquad \begin{array}{ccc} \mathcal{O} & \subseteq & K \\ | & & | \\ \mathbb{Z}_p[\psi] & \subseteq & \mathbb{Q}_p[\psi] \end{array}$$

- There are two types of prime ideals $P \subseteq \Lambda$ of height 1:
  a) $P|(p)$ (only a finite number); Λ/P finite extension of $\mathbb{F}_p[[X]]$.
  b) P is generated by a polynomial not divisible by p; Λ/P finite extension of $\mathbb{Z}_p$.
- From now on, "height 1 prime"="height 1 prime of type b)".
- Consider $\{P_n\}_{n=1}^{\infty}$ a set of distinct height 1 primes of Λ;
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Patching Theorem (Wiles)

Suppose that for each $n \geqslant 1$, there exists a continuous odd representation

$$\varrho_n\colon G_F \to \mathrm{GL}_2(\mathcal{O}_n)$$

unramified outside $\mathfrak{n}p$, for some $\mathfrak{n} \subseteq \mathcal{O}_F$. Suppose that for every prime $\mathfrak{q} \nmid \mathfrak{n}p$, there exist $c_\mathfrak{q}(X), \varepsilon_\mathfrak{q}(X) \in \Lambda$ s.t.

$$\mathrm{Tr}(\varrho_n)(\mathrm{Frob}_\mathfrak{q}) \equiv c_\mathfrak{q}(X) \pmod{P_n},$$
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Then there exists a continuous odd representation $\varrho\colon G_F \to \mathrm{GL}_2(\mathcal{K})$ unramified outside $\mathfrak{n}p$ s.t. for every prime $\mathfrak{q} \nmid \mathfrak{n}p$

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Pseudo-representations

Definition

Let G be a profinite group and let R be a commutative topological integral domain. A *pseudo-representation* of G into R is a triple $\pi = (A_\pi, D_\pi, C_\pi)$ of continuous maps

$$A_\pi\colon G \to R\,, \quad D_\pi\colon G \to R\,, \quad C_\pi\colon G \times G \to R$$

satisfying the following conditions for all elements $g, g_i \in G$:

i) $A_\pi(g_1 g_2) = A_\pi(g_1)A_\pi(g_2) + C_\pi(g_1, g_2)$.

ii) $D_\pi(g_1 g_2) = D_\pi(g_1)D_\pi(g_2) + C_\pi(g_1, g_2)$.

iii) $C(g_1 g_2, g_3) = A_\pi(g_1)C_\pi(g_2, g_3) + D_\pi(g_2)C_\pi(g_1, g_3)$.

iv) $C(g_1, g_2 g_3) = A_\pi(g_3)C_\pi(g_1, g_2) + D_\pi(g_2)C_\pi(g_1, g_3)$.

v) $A_\pi(1) = D_\pi(1) = 1$.

vi) $C_\pi(g, 1) = C_\pi(1, g) = 0$.

vii) $C_\pi(g_1, g_2)C_\pi(g_3, g_4) = C_\pi(g_1, g_4)C_\pi(g_3, g_2)$.

Pseudo-representations vs. Representations

Lemma

$$\left.\begin{array}{l} \varrho\colon G \to \mathrm{GL}_2(R) \textit{ is a rep. s.t.} \\ \varrho(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \pi := (A_\pi(\cdot), D_\pi(\cdot), C_\pi(\cdot,\cdot)) \\ \textit{is a pseudo-rep., where} \\ A_\pi(g) := a(g),\ D_\pi(g) := d(g), \\ C_\pi(g_1, g_2) := b(g_1)c(g_2). \end{array}\right.$$

Conversely, if $\pi = (A_\pi, D_\pi, C_\pi)$ *is a pseudo-representation of* G *into* R *s.t. there exist* $g_1, g_2 \in G$ *with* $C_\pi(g_1, g_2) \in R^*$, *then*

$$\varrho(g) := \begin{pmatrix} A_\pi(g) & C_\pi(g, g_2)/C_\pi(g_1, g_2) \\ C_\pi(g_1, g) & D_\pi(g) \end{pmatrix}$$

is a representation $\varrho\colon G \to \mathrm{GL}_2(R)$.

- If R is a field, then every pseudo-rep. comes from a rep.
- $\mathrm{Tr}(\pi)(g) := A_\pi(g) + D_\pi(g)$, $\quad \det(\pi)(g) := A_\pi(g)D_\pi(g) - C_\pi(g,g)$.

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Odd pseudo-representations

Definition

- A rep. ϱ is *odd* if there exists $\sigma \in G$ of order 2 s.t.

$$\varrho(\sigma) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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Lemma

If $2 \in R^*$, then an odd pseudo-rep. π is determined by $\mathrm{Tr}(\pi)$.

Proof.

$$A_\pi(g) = \frac{\mathrm{Tr}(\pi)(g) - \mathrm{Tr}(\pi)(g\sigma)}{2}, \qquad D_\pi(g) = \frac{\mathrm{Tr}(\pi)(g) + \mathrm{Tr}(\pi)(g\sigma)}{2}.$$

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$$A_\pi(g) = \frac{\mathrm{Tr}(\pi)(g) - \mathrm{Tr}(\pi)(g\sigma)}{2}, \qquad D_\pi(g) = \frac{\mathrm{Tr}(\pi)(g) + \mathrm{Tr}(\pi)(g\sigma)}{2}. \qquad \square$$

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Layout

Λ-adic representations attached to Λ-dic forms

Theorem (Wiles)

For a newform $\mathcal{F} \in \mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}, \psi \,|\, \Lambda)$, *there is a cont. odd irred. rep.*

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unramified outside $\mathfrak{n}p$ *s.t. for every prime* $\mathfrak{q} \nmid \mathfrak{n}p$ *we have*

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- We show:

  1 Eichler-Shimura (k=2)
  2 Patching Theorem
  $\Rightarrow$ Theorem on the previous slide (existence of Galois reps. for $k \geqslant 1$).

- For almost all $r \geqslant 1$

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- $f_r \overset{1}{\rightsquigarrow} \varrho_r \colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}[\zeta_r])$ a continuous irreducible odd representation unramified outside Np satisfying that for every $q \nmid Np$

$$\mathrm{Tr}(\varrho_r)(\mathrm{Frob}_q) = c_q(f_r) \equiv c_q(\mathcal{F})(X) \pmod{P_{2,r}},$$
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- $\{\varrho_r\}_r \overset{2}{\rightsquigarrow} \varrho_\mathcal{F} \colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{K})$.

Warm up: $F = \mathbb{Q}$

- We show:

$$\left.\begin{array}{l} 1 \text{ Eichler-Shimura (k=2)} \\ 2 \text{ Patching Theorem} \end{array}\right\} \Rightarrow \begin{array}{c} \text{Theorem on the previous slide} \\ (\text{existence of Galois reps. for } k \geqslant 1)\,. \end{array}$$

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General case

- Assume $d = [F : \mathbb{Q}]$ is even.
- $\mathfrak{l} \subseteq \mathcal{O}_F$ = a prime s.t $(\mathfrak{l}, \mathfrak{n}p) = 1$.
- Extending coefficients to $\mathcal{K}$. Set

$$\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}, \psi \,|\, \mathcal{K}) := \mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}, \psi \,|\, \Lambda) \otimes_\Lambda \mathcal{K}\,.$$

- Space of oldforms *with respect to* $\mathfrak{l}$:

$$\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}\mathfrak{l}, \psi \,|\, \mathcal{K})^{\mathrm{old}} := \{\mathcal{F}(z) + \mathcal{G}(\mathfrak{l}z) \mid \mathcal{F}, \mathcal{G} \in \mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}, \psi \,|\, \mathcal{K})\}\,.$$

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The congruence module of $\mathcal{F}$

- Set

$$H(\mathcal{F},\mathfrak{l}\,|\,\mathcal{K}) \;:=\; \{\mathcal{H}\in\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}\mathfrak{l},\psi\,|\,\mathcal{K})^{\mathrm{new}}\,|\,\mathcal{H}=\mathcal{G}-u\,\mathcal{F}-v\,\mathcal{F}(\mathfrak{l}z),\ \text{with } \mathcal{G}\in\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}\mathfrak{l},\psi\,|\,\Lambda),\ u,v\in\mathcal{K}\}\,.$$

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  It measures how far the direct sum decomposition over $\mathcal{K}$ fails to be a direct sum decomposition over Λ.

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- There are infinitely many specializations of each $\mathcal{F}_i(\mathfrak{a}_{i,j}z)$ in the special basis satisfying hypothesis *ii*) of Carayol's Theorem at $\mathfrak{l}$ $\rightsquigarrow \varrho_{\mathcal{F}_i(\mathfrak{a}_{i,j}z)}$.
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$$\bigoplus \varrho_{\mathcal{F}_i(\mathfrak{a}_{ij}z)} \otimes \mathcal{K} \colon G_F \to \mathrm{GL}_2(A)$$

- Endow A with an action of $\mathbb{T}$ by transport of structure. The map

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Layout

On the existence of ordinary primes

- For simplicity, take $F = \mathbb{Q}$.
- For $f(q) = \sum_{n\geqslant 1} c_n q^n \in S_k(N,\psi)$, set

$$\Sigma := \{p \text{ prime} \mid c_p \not\equiv 0 \pmod{p}\}.$$

- For general f, is it known whether Σ:
  - has a positive density?
  - contains infinitely many primes?
  - is at least non-empty?
- If f has CM: an affirmative answer is well-known. So assume, from now on, that f does not have CM.
- For $k > 3$: open.
- For $k \leqslant 3$: Σ has a positive density.

Theorem (Serre '81)

The set $S = \{p \text{ prime} \mid c_p = 0\}$ has zero density.

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- Suppose that $k \leqslant 2$. Then, the Ramanujan-Petersson inequality

$$|c_p| \leqslant 2p^{(k-1)/2} \leqslant 2\sqrt{p}$$

  implies that almost all primes not in Σ are in S.
  Serre's theorem $\Rightarrow$ Σ has density 1.

- Suppose now that $k = 3$.
  - ϱ_f is odd $\Rightarrow \mathrm{Tr}(\varrho_f)(\sigma) = 0$.
  - Pick a prime $\ell > 2$. Then, the set

$$S_\ell := \{p \text{ prime} \mid c_p \equiv 0 \pmod{\ell}\}$$

    has a positive density.
  - The Ramanujan-Petersson inequality

$$|c_p| \leqslant 2p$$

    implies now that every prime in S_ℓ and not in Σ is in S.
  - Serre's Theorem $\Rightarrow$ the density of Σ is at least the density of S_ℓ.

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