Galois representations associated to ordinary Hilbert modular forms: Wiles’ theorem

Francesc Fité (Universität Duisburg-Essen)

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Layout

1. Wiles’ result and general notations
2. General strategy of the proof
3. Tools for the proof
4. Sketch of the proof
5. Are there any ordinary primes?
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Notations

- $F$ a totally real field; $\mathcal{O}_F$ its ring of integers; $d := [F : \mathbb{Q}]$.
- $\mathfrak{n} \subseteq \mathcal{O}_F$ integral ideal; $\psi_0 : (\mathcal{O}_F/\mathfrak{n})^* \rightarrow \overline{\mathbb{Q}}^*$; $\mathfrak{a}$ a fractional ideal.
- $S_k(\Gamma(\mathfrak{a}, \mathfrak{n}), \psi_0) = \text{space of Hilbert cusp forms } f : \mathcal{H}^d \rightarrow \mathbb{C} \text{ of parallel weight } k \geq 1, \text{ character } \psi_0, \text{ and relative to } \Gamma(\mathfrak{a}, \mathfrak{n})$.
- $h = \text{strict class number of } F$ (t$_{\gamma}$ reps. of the strict class ideals).
- $S_k(\mathfrak{n}, \psi_0)$ denotes the space of $f := (f_1, \ldots, f_h) \in \prod_{\gamma=1}^{h} S_k(\Gamma(t_{\gamma}d, \mathfrak{n}), \psi_0)$. 

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\]
Dirichlet series

- Each $f_\gamma$ admits a Fourier expansion

$$f_\gamma(z_1, \ldots, z_d) = \sum_{0 \ll \mu \in t_\gamma} a_\gamma(\mu) e^{2\pi i (\sum_{j=1}^d \mu_j z_j)}, \quad \text{for } (z_1, \ldots, z_d) \in \mathcal{H}^d.$$  

($\mu_1, \ldots, \mu_d$ denote the images of $\mu$ by the $d$ embeddings of $F$ into $\mathbb{C}$).

- For $0 \neq \alpha \subseteq \mathcal{O}_F$, there exist $\gamma \in \{1, \ldots, h\}$ and a totally positive $\mu \in t_\gamma$ such that $\alpha = \mu t_\gamma^{-1}$. Define

$$c(\alpha, f) := a_\gamma(\mu) N(t_\gamma)^{-k/2}.$$  

(it depends neither on the choice of $\gamma$ nor of $t_\gamma$).

- The Dirichlet series associated to $f$ is

$$D(f, s) := \sum_{0 \neq \alpha \subseteq \mathcal{O}_F} c(\alpha, f) N(\alpha)^{-s}.$$
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Ordinary primes

- There is a theory of Hecke operators \( \{ T_n(a), S_n(a) \}_{a \in \mathcal{O}_F} \) on \( S_k(n, \psi_0) \).

- \( \psi_0 \mapsto \psi: \mathcal{I}_{n\infty} \to \mathbb{Q}^* \) ray class char. of modulus \( n\infty \) restricting to \( \psi_0 \) on \( (\mathcal{O}_F/n)^* \). Set

\[
S_k(n, \psi) := \{ f \in S_k(n, \psi_0) \mid S_n(a)(f) = \psi(a)f \text{ for all } a \subseteq \mathcal{O}_F \}.
\]

- For \( f \in S_k(n, \psi) \) a newform, set

\[
K_f := \mathbb{Q}(\{ c(a, f) \}_{a \in \mathcal{O}_F}); \quad \mathcal{O}_f \text{ its ring of integers.}
\]

- \( \lambda \subseteq \mathcal{O}_f \) prime. \( f \) is ordinary at \( \lambda \) if for \( p \subseteq \mathcal{O}_F, p|N(\lambda), \)

\[
x^2 - c(p, f)x + \psi(p)N(p)^{k-1}
\]

has a unit root mod \( \lambda \).
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  has a unit root mod \( \lambda \).
Main Theorem (Wiles)

If $f$ is ordinary at $\lambda$, there exists a continuous odd irreducible rep.

$$\rho_{f,\lambda} : G_F \to \text{GL}_2(\mathcal{O}_{f,\lambda})$$

unramified outside $\mathfrak{n}\mathcal{N}(\lambda)$ and such that for all primes $q \nmid \mathfrak{n}\mathcal{N}(\lambda)$

$$\text{Tr}(\rho_{f,\lambda})(\text{Frob}_q) = c(q, f),$$

$$\det(\rho_{f,\lambda})(\text{Frob}_q) = \psi(q)\mathcal{N}(q)^{k-1}.$$
Wiles’ result

Main Theorem (Wiles)

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- In the previous talk we saw:

Theorem (Carayol)

For \( k \geq 2 \) and \( f \) not necessarily ordinary at \( \lambda \), there exists \( \varrho_{f,\lambda} \) if either

i) \( d \) is odd; or

ii) \( d \) is even and there is a prime \( p \mid \mid \mathfrak{n} \).
Wiles’ result and general notations

General strategy of the proof

Tools for the proof

Sketch of the proof

Are there any ordinary primes?
General strategy

- $S_k(n, \psi \mid \mathbb{Z}[\psi]) := \{g \in S_k(n, \psi) \mid c(a, g) \in \mathbb{Z}[\psi] \text{ for all } a \subseteq \mathcal{O}_F\}$.
- Fix a prime $p$ from now on. For simplicity assume $p \geq 3$.
- For a subring $\mathbb{Z}[\psi] \subseteq A \subseteq \overline{\mathbb{Q}}_p$, set
  \[ S_k(n, \psi \mid A) := S_k(n, \psi \mid \mathbb{Z}[\psi]) \otimes_{\mathbb{Z}[\psi]} A \]

Step 1: $\Lambda$-adic forms

Define $S(\overline{n}, \psi \mid \Lambda)$ and specialization maps $\nu_{k,r} : \Lambda \to \overline{\mathbb{Q}}_p$ such that

\[ \mathcal{F} \in S(\overline{n}, \psi \mid \Lambda) \Rightarrow \nu_{k,r}(\mathcal{F}) \in S_k(np^r, \psi_{\zeta_{p^r}}, \omega^{2-k} \mid \mathcal{O}[\zeta_{p^r}]) . \]
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  \]
- \( \mathcal{K} \) a finite extension of \( \mathbb{Q}_p((X)) \); \( \Lambda = \) integral closure of \( \mathbb{Z}_p[[X]] \).
  Assume \( \Lambda \supseteq \mathbb{Z}_p[\psi][[X]] \).
- \( \mathcal{K} := \mathcal{K} \cap \overline{\mathbb{Q}}_p \); \( \mathcal{O} \) its ring of integers.

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- \( S_k(n, \psi | \mathbb{Z}[\psi]) := \{ g \in S_k(n, \psi) \mid c(a, g) \in \mathbb{Z}[\psi] \text{ for all } a \subseteq \mathcal{O}_F \} \).
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General strategy

Step 2: Lifting

Define $S^\text{ord}_k(n, \psi | O) \subseteq S_k(n, \psi | O)$ (containing the $f$ ordinary at $\lambda|p$) s.t.

$$f \in S^\text{ord}_k(n, \psi \omega^{2-k} | O) \Rightarrow \exists \mathcal{F} \in S^\text{ord}(\bar{n}, \psi \mid \Lambda) \text{ and } \nu_{k,0} \text{ s.t. } \nu_{k,0}(\mathcal{F}) = f.$$

Step 3: Patching – Theory of pseudo-representations

Write $f_{k,r} := \nu_{k,r}(\mathcal{F})$.

There exists $\varrho_{f_{k,r},\lambda}$ for infinitely many $\nu_{k,r}$

$$\Rightarrow \text{ There exists } \varrho_\mathcal{F} : G_F \to GL_2(\mathcal{K}) \text{ s.t. } \nu_{k,r}(\varrho_\mathcal{F}) = \varrho_{f_{k,r},\lambda} \text{ for almost every } \nu_{k,\zeta}.$$

$$f \text{ ordinary at } \lambda \quad \xrightarrow{\text{Step 2}} \quad \exists \mathcal{F} \text{ s.t. } \nu(\mathcal{F}) = f \quad \xrightarrow{\text{Step 3}}$$

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Define $S_{k}^{\text{ord}}(n, \psi \mid \mathcal{O}) \subseteq S_{k}(n, \psi \mid \mathcal{O})$ (containing the $f$ ordinary at $\lambda \mid p$) s.t.

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\( f \) ordinary at \( \lambda \) \( \implies \exists \mathcal{F} \) s.t. \( \nu(\mathcal{F}) = f \)

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Wiles’ result and general notations

General strategy of the proof

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Sketch of the proof

Are there any ordinary primes?
Step 1: Defining $\Lambda$-adic forms

- Recall that $p \geq 3$ and further assume $[\mathbb{Q}_\infty \cap F : \mathbb{Q}] = 1$.
- For $k \geq 1$, $r \geq 0$, define the specialization map

$$\nu_{k,r} : \mathbb{Z}_p[[X]] \rightarrow \mathbb{Z}_p[\zeta], \quad X \mapsto \zeta(1 + p)^{k-2} - 1,$$

where $\zeta^r = 1$.
- By the going-up theorem:

$$P_{k,r} \subseteq \Lambda \subseteq \mathcal{K}$$

$$\ker(\nu_{k,r}) \subseteq \mathbb{Z}_p[[X]] \subseteq \mathbb{Q}_p((X))$$

- Recall that $K = \mathcal{K} \cap \overline{\mathbb{Q}}_p$ and $\mathcal{O} = \Lambda \cap \overline{\mathbb{Q}}_p$.
- We may view $P_{k,r}$ as the kernel of an $\mathcal{O}$-algebra homomorphism $\nu_{k,r} : \Lambda \rightarrow \overline{\mathbb{Q}}_p$ extending $\nu_{k,r}$. 
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  \[
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  P_{k,r} \\ \downarrow \ker(\nu_{k,r}) \\
  \end{array}
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  \]

  \[
  \begin{array}{c}
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  \end{array}
  \leq \mathbb{Q}_p((X))
  \]

- Recall that $K = \mathcal{K} \cap \overline{\mathbb{Q}}_p$ and $\mathcal{O} = \Lambda \cap \overline{\mathbb{Q}}_p$.
- We may view $P_{k,r}$ as the kernel of an $\mathcal{O}$-algebra homomorphism $\nu_{k,r} : \Lambda \to \overline{\mathbb{Q}}_p$ extending $\nu_{k,r}$. 
For a fractional ideal $a$ of $F$ s.t. $(a, p) = 1$, we can write

$$N(a) = (1 + p)^\alpha \delta,$$

with $\delta \in \mu_{p-1}$, $\alpha \in \mathbb{Z}_p$.

Given $\psi: \mathbb{Q}_p^\infty \to \overline{\mathbb{Q}}^*$ and $\zeta^p = 1$, define

$$\psi: \lim_{\rightarrow} l_{np^t} \to \Lambda, \quad \psi(a) = \psi(a)(1 + X)^\alpha,$$

$$\varrho_\zeta: l_{p^r} \mathcal{O}_F \to \overline{\mathbb{Q}}^*, \quad \varrho_\zeta(a) = \zeta^\alpha,$$

$$\omega: l_{p \mathcal{O}_F} \to \overline{\mathbb{Q}}^*, \quad \omega(a) = N(a)/(1 + p)^\alpha = \delta.$$

We will call $\omega$ the Teichmüller character.
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Given $\psi : I_{n\infty} \to \overline{\mathbb{Q}}^*$ and $\zeta^p = 1$, define

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We will call $\omega$ the Teichmüller character.
Definition

A \( \Lambda \)-adic cuspidal form \( \mathcal{F} \) over \( F \) of level \( n \) and character \( \psi: \lim_{\mathcal{I} \rightarrow \Lambda} l_{np^t} \rightarrow \Lambda \) is a collection of elements of \( \Lambda \)

\[
\{c(a, \mathcal{F})(X)\}_{0 \neq a \subseteq \mathcal{O}_F} \subseteq \Lambda,
\]

such that, for almost every \( \nu_{k,r} \), with \( k \geq 2, r \geq 0 \), there exists

\[
f_{\nu_{k,r}} \in S_k(n p^r, \psi \zeta \omega^{2-k} \mid \mathcal{O}[\zeta])
\]

whose associated Dirichlet series is

\[
D(f_{\nu_{k,r}}, s) = \sum_{0 \neq a \subseteq \mathcal{O}_F} \nu_{k,r}(c(a, \mathcal{F})(X)) N(a)^{-s}.
\]

- By abuse of notation, write \( \nu_{k,r}(\mathcal{F}) := f_{\nu_{k,r}} \).
- \( S(n, \psi \mid \Lambda) \) is the \( \Lambda \)-module of \( \Lambda \)-adic cusp forms.
- Set \( S(\mathcal{O}, \psi \mid \Lambda) = \bigcup_{t=0}^{\infty} S(np^t, \psi \mid \Lambda) \).
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Francesc Fité (Universität Duisburg-Essen)
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Step 2: The space of classical $p$-stabilized forms

- We work at level $np^r$, with $r \geq 1$.
- The Hida operator

\[ e := \lim_{n \to \infty} T_{np^r}(p)^n! : S_k(np^r, \psi | \mathcal{O}) \to S_k(np^r, \psi | \mathcal{O}). \]

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- $\mathfrak{P} := \text{product of primes of } \mathcal{O}_F \text{ above } p \text{ not dividing } m$.
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Example

- $F = \mathbb{Q}$ and $N \geq 1$ with $(N, p) = 1$.
- Let $f = \sum_{n \geq 1} c_n q^n \in S_k(\Gamma_0(N), \psi)$ be an ordinary eigenform.
- $x^2 - c_p(f)x + \psi(p)p^{k-1}$ has roots $\alpha, \beta$ and suppose that $\alpha$ is a unit.
- Using usual relations between Hecke operators one can check that $T_{Np}(p)$ acts on $\langle f, f(pz) \rangle$ by means of

$$B = \begin{pmatrix} c_p & 1 \\ -\psi(p)p^{k-1} & 0 \end{pmatrix}.$$

- $B$ has eigenvectors $f_\alpha(z) := f(z) - \beta f(pz)$, $f_\beta(z) := f(z) - \alpha f(pz)$:

$$T_{Np}(p)(f_\alpha) = \alpha f_\alpha, \quad T_{Np}(p)(f_\beta) = \beta f_\beta$$
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The Lifting Theorem

- **The space of $p$-stabilized $\Lambda$-adic forms**: There is an idempotent

$$E : S(\overline{n}, \psi | \Lambda) \to S(\overline{n}, \psi | \Lambda)$$

of $\text{End}_\Lambda(S(\overline{n}, \psi | \Lambda))$ s.t. for almost every $\nu$ we have

$$\nu(E(F)) = e(\nu(F)).$$

---

**Theorem (Wiles, helped by Taylor & Shimura)**

$S^{\text{ord}}(\overline{n}, \psi | \Lambda) := E S(\overline{n}, \psi | \Lambda)$ is free $\Lambda$-module of finite rank.

- **Hecke operators**: For every $a \subseteq \mathcal{O}_F$, one can define $\Lambda$-linear maps

$$T(a) : S(\overline{n}, \psi | \Lambda) \to S(\overline{n}, \psi | \Lambda)$$

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The lifting theorem

- $\mathcal{F} \in S(\overline{n}, \psi | \Lambda)$ is called:
  - a Hecke eigenform if it is an eigenvector for $T(a)(\mathcal{F})$ for every $a \subseteq \mathcal{O}_F$.
  - normalized if $c(\mathcal{O}_F, \mathcal{F})(X) = 1$.
  - a newform if it is a normalized eigenform s.t. $\nu(\mathcal{F})$ is a newform of level divisible by $n_0$ (with $(n_0, p) = 1$) for almost every $\nu$.
- In fact, $\mathcal{F}$ is a newform $\iff \nu(\mathcal{F})$ is a newform for infinitely many $\nu$.

Theorem (Hida for $F = \mathbb{Q}$ and $k \geq 2$; Wiles in general)

For a newform $f \in S^\text{ord}_k(n, \psi \omega^{2-k} | \mathcal{O})$, $k \geq 1$, there exist $\Lambda \supseteq \mathbb{Z}_p[[X]]$, $\nu_{k,0}$, and a newform $\mathcal{F} \in S^\text{ord}(\overline{n}, \psi | \Lambda)$ s.t. $\nu_{k,0}(\mathcal{F}) = f$. 

Francesc Fité (Universität Duisburg-Essen)
The lifting theorem

- \( \mathcal{F} \in S(\overline{n}, \psi | \Lambda) \) is called:
  - a Hecke eigenform if it is an eigenvector for \( T(a)(\mathcal{F}) \) for every \( a \subseteq \mathcal{O}_F \).
  - normalized if \( c(\mathcal{O}_F, \mathcal{F})(X) = 1 \).
  - a newform if it is a normalized eigenform s.t. \( \nu(\mathcal{F}) \) is a newform of level divisible by \( n_0 \) (with \( (n_0, p) = 1 \)) for almost every \( \nu \).
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The lifting theorem

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- $\mathcal{F} \in S(\bar{n}, \psi \mid \Lambda)$ is called:
  - a \textit{Hecke eigenform} if it is an eigenvector for $T(\alpha)(\mathcal{F})$ for every $\alpha \subseteq \mathcal{O}_F$.
  - \textit{normalized} if $c(\mathcal{O}_F, \mathcal{F})(X) = 1$.
  - a \textit{newform} if it is a normalized eigenform s.t. $\nu(\mathcal{F})$ is a newform of level divisible by $n_0$ (with $(n_0, p) = 1$) for almost every $\nu$.

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\begin{itemize}
\item Theorem (Hida for $F = \mathbb{Q}$ and $k \geq 2$; Wiles in general)
\end{itemize}

For a newform $f \in S^\text{ord}_k(n, \psi \omega^{2-k} \mid \mathcal{O})$, $k \geq 1$, there exist $\Lambda \supseteq \mathbb{Z}_p[[X]]$, $\nu_{k,0}$, and a newform $\mathcal{F} \in S^\text{ord}(\bar{n}, \psi \mid \Lambda)$ s.t. $\nu_{k,0}(\mathcal{F}) = f$. 
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**Theorem (Hida for $F = \mathbb{Q}$ and $k \geq 2$; Wiles in general)**

*For a newform $f \in S^\text{ord}_k(n, \psi \omega^{2-k} \mid \mathcal{O}), k \geq 1$, there exist $\Lambda \supseteq \mathbb{Z}_p[[X]]$, $\nu_{k,0}$, and a newform $\mathcal{F} \in S^\text{ord}(\overline{n}, \psi \mid \Lambda)$ s.t. $\nu_{k,0}(\mathcal{F}) = f$.***
Step 3: Patching

- Recall the setting

\[ \Lambda \subseteq \mathcal{K} \]
\[ \mathbb{Z}_p[\psi][[X]] \subseteq \mathbb{Q}_p[\psi]((X)) \]
\[ \mathcal{O} \subseteq \mathcal{K} \]
\[ \mathbb{Z}_p[\psi] \subseteq \mathbb{Q}_p[\psi] \]

- There are two types of prime ideals \( P \subseteq \Lambda \) of height 1:
  a) \( P|\langle p \rangle \) (only a finite number); \( \Lambda/P \) finite extension of \( \mathbb{F}_p[[X]] \).
  b) \( P \) is generated by a polynomial not divisible by \( p \); \( \Lambda/P \) finite extension of \( \mathbb{Z}_p \).

- From now on, “height 1 prime” = “height 1 prime of type b)”.

- Consider \( \{P_n\}_{n=1}^{\infty} \) a set of distinct height 1 primes of \( \Lambda \);
- \( K_n = \) field of fractions of \( \Lambda/P_n \);
- \( \mathcal{O}_n = \) integral closure of \( \Lambda/P_n \) in \( K_n \).
Step 3: Patching

- Recall the setting

\[
\begin{array}{ccc}
\Lambda & \subseteq & K \\
\mid & & \mid \\
\mathbb{Z}_p[\psi][[X]] & \subseteq & \mathbb{Q}_p[\psi](X) \\
\mid & & \mid \\
\mathbb{Z}_p[\psi] & \subseteq & \mathbb{Q}_p[\psi]
\end{array}
\]

- There are two types of prime ideals \( P \subseteq \Lambda \) of height 1:
  a) \( P | (p) \) (only a finite number); \( \Lambda/P \) finite extension of \( \mathbb{F}_p[[X]] \).
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Step 3: Patching

- Recall the setting

\[
\begin{align*}
\Lambda & \subseteq K \\
\mathbb{Z}_p[\psi][[X]] & \subseteq \mathbb{Q}_p[\psi]((X)) \\
\mathcal{O} & \subseteq K \\
\mathbb{Z}_p[\psi] & \subseteq \mathbb{Q}_p[\psi]
\end{align*}
\]

- There are two types of prime ideals \( P \subseteq \Lambda \) of height 1:
  a) \( P|_p(p) \) (only a finite number); \( \Lambda/P \) finite extension of \( \mathbb{F}_p[[X]] \).
  b) \( P \) is generated by a polynomial not divisible by \( p \); \( \Lambda/P \) finite extension of \( \mathbb{Z}_p \).

- From now on, “height 1 prime” = “height 1 prime of type b)”.

- Consider \( \{P_n\}_{n=1}^{\infty} \) a set of distinct height 1 primes of \( \Lambda \);
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Patching Theorem (Wiles)

Suppose that for each \( n \geq 1 \), there exists a continuous odd representation

\[ \varrho_n : G_F \to \text{GL}_2(\mathcal{O}_n) \]

unramified outside \( np \), for some \( n \subseteq \mathcal{O}_F \). Suppose that for every prime \( q \nmid np \), there exist \( c_q(X), \varepsilon_q(X) \in \Lambda \) s.t.

\[
\begin{align*}
\text{Tr}(\varrho_n)(\text{Frob}_q) &\equiv c_q(X) \pmod{P_n}, \\
\text{det}(\varrho_n)(\text{Frob}_q) &\equiv \varepsilon_q(X) \pmod{P_n}.
\end{align*}
\]

Then there exists a continuous odd representation \( \varrho : G_F \to \text{GL}_2(\mathcal{K}) \) unramified outside \( np \) s.t. for every prime \( q \nmid np \)

\[
\begin{align*}
\text{Tr}(\varrho)(\text{Frob}_q) &= c_q(X) \in \Lambda, \\
\text{det}(\varrho)(\text{Frob}_q) &= \varepsilon_q(X) \in \Lambda.
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Suppose that for each $n \geq 1$, there exists a continuous odd representation

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$$\text{Tr}(\varrho_n)(\text{Frob}_q) \equiv c_q(X) \pmod{P_n},$$
$$\text{det}(\varrho_n)(\text{Frob}_q) \equiv \varepsilon_q(X) \pmod{P_n}.$$

Then there exists a continuous odd representation $\varrho : G_F \to \text{GL}_2(\mathcal{K})$ unramified outside $np$ s.t. for every prime $q \nmid np$

$$\text{Tr}(\varrho)(\text{Frob}_q) = c_q(X) \in \Lambda,$$
$$\text{det}(\varrho)(\text{Frob}_q) = \varepsilon_q(X) \in \Lambda.$$
Patching Theorem (Wiles)

Suppose that for each \( n \geq 1 \), there exists a continuous odd representation

\[
\varrho_n : G_F \to \text{GL}_2(\mathcal{O}_n)
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unramified outside \( np \), for some \( n \subseteq \mathcal{O}_F \). Suppose that for every prime \( q \nmid np \), there exist \( c_q(X), \varepsilon_q(X) \in \Lambda \) s.t.

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\text{Tr}(\varrho_n)(\text{Frob}_q) \equiv c_q(X) \pmod{P_n},
\]

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\det(\varrho_n)(\text{Frob}_q) \equiv \varepsilon_q(X) \pmod{P_n}.
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\]

\[
\det(\varrho)(\text{Frob}_q) = \varepsilon_q(X) \in \Lambda.
\]
Pseudo-representations

**Definition**

Let $G$ be a profinite group and let $R$ be a commutative topological integral domain. A *pseudo-representation* of $G$ into $R$ is a triple $\pi = (A_\pi, D_\pi, C_\pi)$ of continuous maps

$$A_\pi : G \to R, \quad D_\pi : G \to R, \quad C_\pi : G \times G \to R$$

satisfying the following conditions for all elements $g, g_i \in G$:

i) $A_\pi(g_1 g_2) = A_\pi(g_1) A_\pi(g_2) + C_\pi(g_1, g_2)$.

ii) $D_\pi(g_1 g_2) = D_\pi(g_1) D_\pi(g_2) + C_\pi(g_1, g_2)$.

iii) $C(g_1 g_2, g_3) = A_\pi(g_1) C_\pi(g_2, g_3) + D_\pi(g_2) C_\pi(g_1, g_3)$.

iv) $C(g_1, g_2 g_3) = A_\pi(g_3) C_\pi(g_1, g_2) + D_\pi(g_2) C_\pi(g_1, g_3)$.

v) $A_\pi(1) = D_\pi(1) = 1$.

vi) $C_\pi(g, 1) = C_\pi(1, g) = 0$.

vii) $C_\pi(g_1, g_2) C_\pi(g_3, g_4) = C_\pi(g_1, g_4) C_\pi(g_3, g_2)$.
Pseudo-representations vs. Representations

Lemma

\[ \varrho : G \to \text{GL}_2(R) \text{ is a rep. s.t.} \]
\[ \varrho(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix} \]
\[ \implies \begin{cases} \pi := (A_\pi(\cdot), D_\pi(\cdot), C_\pi(\cdot, \cdot)) \\ \text{is a pseudo-rep., where} \\ A_\pi(g) := a(g), \ D_\pi(g) := d(g), \\ C_\pi(g_1, g_2) := b(g_1)c(g_2). \end{cases} \]

Conversely, if \( \pi = (A_\pi, D_\pi, C_\pi) \) is a pseudo-representation of \( G \) into \( R \) s.t.
there exist \( g_1, g_2 \in G \) with \( C_\pi(g_1, g_2) \in R^* \), then

\[ \varrho(g) := \begin{pmatrix} A_\pi(g) & C_\pi(g, g_2)/C_\pi(g_1, g_2) \\ C_\pi(g_1, g) & D_\pi(g) \end{pmatrix} \]

is a representation \( \varrho : G \to \text{GL}_2(R) \).

- If \( R \) is a field, then every pseudo-rep. comes from a rep.
- \( \text{Tr}(\pi)(g) := A_\pi(g) + D_\pi(g), \quad \det(\pi)(g) := A_\pi(g)D_\pi(g) - C_\pi(g, g). \)
Lemma

\( \varrho : G \to \text{GL}_2(R) \) is a rep. s.t.
\[
\varrho(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}
\]

\[\Rightarrow\]
\[
\pi := (A_\pi(\cdot), D_\pi(\cdot), C_\pi(\cdot, \cdot))
\]
is a pseudo-rep., where
\[
A_\pi(g) := a(g), \quad D_\pi(g) := d(g),
\]
\[
C_\pi(g_1, g_2) := b(g_1)c(g_2).
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\[ \varrho : G \rightarrow \text{GL}_2(R) \text{ is a rep. s.t.} \]
\[ \varrho(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix} \Rightarrow \left\{ \begin{array}{l}
\pi := (A_\pi(\cdot), D_\pi(\cdot), C_\pi(\cdot, \cdot)) \\
is a pseudo-rep., where \\
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\end{array} \right. \]

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Odd pseudo-representations

Definition

- A rep. \( \varrho \) is *odd* if there exists \( \sigma \in G \) of order 2 s.t.
  \[
  \varrho(\sigma) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
  \]

- A pseudo-rep. \( \pi \) is *odd* if there exists \( \sigma \in G \) of order 2 s.t.
  \[
  A_\pi(\sigma) = -1, \quad D_\pi(\sigma) = 1, \quad C_\pi(g, \sigma) = C_\pi(\sigma, g) = 0 \quad \forall g \in G.
  \]

Lemma

If \( 2 \in R^* \), then an odd pseudo-rep. \( \pi \) is determined by \( \text{Tr}(\pi) \).

Proof.

\[
A_\pi(g) = \frac{\text{Tr}(\pi)(g) - \text{Tr}(\pi)(g\sigma)}{2}, \quad D_\pi(g) = \frac{\text{Tr}(\pi)(g) + \text{Tr}(\pi)(g\sigma)}{2}.
\]
Odd pseudo-representations

Definition

- A rep. $\varrho$ is **odd** if there exists $\sigma \in G$ of order 2 s.t.
  \[ \varrho(\sigma) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \]

- A pseudo-rep. $\pi$ is **odd** if there exists $\sigma \in G$ of order 2 s.t.
  \[ A_\pi(\sigma) = -1, \quad D_\pi(\sigma) = 1, \quad C_\pi(g, \sigma) = C_\pi(\sigma, g) = 0 \quad \forall g \in G. \]

Lemma

*If* $2 \in \mathbb{R}^*$, *then* an odd pseudo-rep. $\pi$ *is determined by* $\text{Tr}(\pi)$.

Proof.

\[ A_\pi(g) = \frac{\text{Tr}(\pi)(g) - \text{Tr}(\pi)(g \sigma)}{2}, \quad D_\pi(g) = \frac{\text{Tr}(\pi)(g) + \text{Tr}(\pi)(g \sigma)}{2}. \]
Odd pseudo-representations

**Definition**

- A rep. $\varrho$ is *odd* if there exists $\sigma \in G$ of order 2 s.t.
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**Lemma**

*If* $2 \in R^*$, *then an odd pseudo-rep. $\pi$ is determined by $\text{Tr}(\pi)$.***

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\[A_\pi(g) = \frac{\text{Tr}(\pi)(g) - \text{Tr}(\pi)(g\sigma)}{2}, \quad D_\pi(g) = \frac{\text{Tr}(\pi)(g) + \text{Tr}(\pi)(g\sigma)}{2}.\]
Proof of the Patching Theorem

- odd $\varrho_n : G_F \to GL_2(O_n) \rightsquigarrow$ odd $\pi_n$ with values in $O_n$
- $\pi_n$ is determined by $\text{Tr}(\pi_n) = \text{Tr}(\varrho_n) \in \Lambda/P_n$
- $\varrho_n$ is with values in $\Lambda/P_n$.

Write $Q_r = P_1 \cap \cdots \cap P_r$.

Suppose we have constructed a pseudo-rep. $\alpha_r$ in $\Lambda/Q_r$ s.t.

$$\alpha_r \equiv \pi_n \pmod{P_n} \quad \text{for } 1 \leq n \leq r.$$

Observe that for $1 \leq n \leq r$

$$\text{Tr}(\alpha_r) \equiv \text{Tr}(\pi_n) \pmod{(P_n, P_{r+1})}$$
Proof of the Patching Theorem

- odd $\varrho_n: G_F \to \text{GL}_2(\mathcal{O}_n)$ \(\sim\) odd $\pi_n$ with values in $\mathcal{O}_n$

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  \]

- Observe that for $1 \leq n \leq r$

  \[
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  \]
Proof of the Patching Theorem

- odd \( \varrho_n : G_F \rightarrow \text{GL}_2(\mathcal{O}_n) \quad \sim \quad \text{odd } \pi_n \text{ with values in } \mathcal{O}_n 

- \( \pi_n \) is determined by \( \text{Tr}(\pi_n) = \text{Tr}(\varrho_n) \in \Lambda/P_n \quad \sim \quad \pi_n \) is with values in \( \Lambda/P_n \).

- Write \( Q_r = P_1 \cap \cdots \cap P_r \).

- Suppose we have constructed a pseudo-rep. \( \alpha_r \) in \( \Lambda/Q_r \) s.t.
  \[
  \alpha_r \equiv \pi_n \pmod{P_n} \quad \text{for } 1 \leq n \leq r .
  \]

- Observe that for \( 1 \leq n \leq r \)
  \[
  \text{Tr}(\alpha_r) \equiv \text{Tr}(\pi_n) = \text{Tr}(\varrho_n) = \text{Tr}((\varrho_n)^{p-1}) = \text{Tr}(\pi_{n+1}) \pmod{(P_n, P_{r+1})}, 
  \]
  \[
  \Rightarrow \text{Tr}(\alpha_r) \equiv \text{Tr}(\pi_{n+1}) \pmod{(Q_r, P_{r+1})}, 
  \]
  \[
  \Rightarrow \alpha_r \equiv \pi_{n+1} \pmod{(Q_r, P_{r+1})}. 
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Proof of the Patching Theorem

- odd $\varrho_n : G_F \to \text{GL}_2(\mathcal{O}_n) \rightsquigarrow$ odd $\pi_n$ with values in $\mathcal{O}_n$
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- Write $Q_r = P_1 \cap \cdots \cap P_r$.
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  $\alpha_r \equiv \pi_n \pmod{P_n}$ for $1 \leq n \leq r$.

Observe that for $1 \leq n \leq r$

$$\text{Tr}(\alpha_r) \equiv \text{Tr}(\pi_n) = \text{Tr}(\varrho_n) = \text{Tr}(\pi_r + 1) \equiv \text{Tr}(\pi_r) \pmod{(P_n, P_{r+1})}$$

$$\Rightarrow \text{Tr}(\alpha_r) = \text{Tr}(\pi_r) \pmod{(Q_r, P_{r+1})}$$

$$\Rightarrow \pi_r \equiv \pi_r \pmod{(Q_r, P_{r+1})}$$
Proof of the Patching Theorem

- odd $\varrho_n: G_F \to \text{GL}_2(\mathcal{O}_n)$ $\rightsquigarrow$ odd $\pi_n$ with values in $\mathcal{O}_n$
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  \[ \alpha_r \equiv \pi_n \pmod{P_n} \quad \text{for} \quad 1 \leq n \leq r. \]
- Observe that for $1 \leq n \leq r$
  \[ \text{Tr}(\alpha_r) \equiv \text{Tr}(\pi_n) = \text{Tr}(\varrho_n) = \text{Tr}(\varrho_{r+1}) = \text{Tr}(\pi_{r+1}) \pmod{(P_n, P_{r+1})} \]
  \[ \Rightarrow \text{Tr}(\alpha_r) \equiv \text{Tr}(\pi_{r+1}) \pmod{(Q_r, P_{r+1})} \]
  \[ \Rightarrow n \equiv \pi_{r+1} \pmod{(Q_r, P_{r+1})}. \]
Proof of the Patching Theorem

- odd $\varrho_n : G_F \rightarrow \text{GL}_2(\mathcal{O}_n) \rightsquigarrow$ odd $\pi_n$ with values in $\mathcal{O}_n$
- $\pi_n$ is determined by $\text{Tr}(\pi_n) = \text{Tr}(\varrho_n) \in \Lambda/P_n$
  $\rightsquigarrow \pi_n$ is with values in $\Lambda/P_n$.
- Write $Q_r = P_1 \cap \cdots \cap P_r$.
- Suppose we have constructed a pseudo-rep. $\alpha_r$ in $\Lambda/Q_r$ s.t.
  \[ \alpha_r \equiv \pi_n \pmod{P_n} \quad \text{for } 1 \leq n \leq r. \]
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  \[ \Rightarrow \text{Tr}(\alpha_r) \equiv \text{Tr}(\pi_{r+1}) \pmod{(Q_r, P_{r+1})} \]

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$$\alpha_r \equiv \pi_n \pmod{P_n} \quad \text{for } 1 \leq n \leq r.$$  

- Observe that for $1 \leq n \leq r$

$$\text{Tr}(\alpha_r) \equiv \text{Tr}(\pi_n) = \text{Tr}(\varrho_n) \equiv \text{Tr}(\varrho_{r+1}) = \text{Tr}(\pi_{r+1}) \pmod{(P_n, P_{r+1})}$$

$$\Rightarrow \text{Tr}(\alpha_r) \equiv \text{Tr}(\pi_{r+1}) \pmod{(Q_r, P_{r+1})}$$

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Proof of the Patching Theorem

- odd $\varrho_n : G_F \to \text{GL}_2(\mathcal{O}_n) \overset{\sim}{\to} \text{odd } \pi_n$ with values in $\mathcal{O}_n$

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Proof of the Patching Theorem

- odd $\varrho_n : G_F \rightarrow GL_2(\mathcal{O}_n) \leadsto$ odd $\pi_n$ with values in $\mathcal{O}_n$
- $\pi_n$ is determined by $\text{Tr}(\pi_n) = \text{Tr}(\varrho_n) \in \Lambda/P_n$
- $\leadsto \pi_n$ is with values in $\Lambda/P_n$.
- Write $Q_r = P_1 \cap \cdots \cap P_r$.
- Suppose we have constructed a pseudo-rep. $\alpha_r$ in $\Lambda/Q_r$ s.t.

$$\alpha_r \equiv \pi_n \pmod{P_n} \quad \text{for } 1 \leq n \leq r.$$ 

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$$\text{Tr}(\alpha_r) \equiv \text{Tr}(\pi_n) = \text{Tr}(\varrho_n) \equiv \text{Tr}(\varrho_{r+1}) = \text{Tr}(\pi_{r+1}) \pmod{(P_n, P_{r+1})}$$

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Proof of the Patching Theorem.

By

$$0 \to \Lambda/Q_{r+1} \to \Lambda/Q_r \oplus \Lambda/P_{r+1} \to \Lambda/(Q_r, P_{r+1}) \to 0,$$

we may lift the pseudo-rep. \( \alpha_r \oplus \pi_{r+1} \) into \( \Lambda/Q_r \oplus \Lambda/P_{r+1} \) to a pseudo-rep. \( \alpha_{r+1} \) into \( \Lambda/Q_{r+1} \) s.t.

\[
\alpha_{r+1} \equiv \pi_n \pmod{P_n} \quad \text{for } 1 \leq n \leq r + 1.
\]

Set \( \alpha := \lim_{\leftarrow} \alpha_n \).

\( \alpha \) is a pseudo-rep. into \( \lim_{\leftarrow} \Lambda/P_n \cong \Lambda \), since \( \cap_{n=1}^{\infty} P_n = 0 \).

\( \alpha \) pseudo-rep. into \( K \) into a rep. \( \rho \colon G_r \to \GL_3(K) \), which has the desired properties.
Proof of the Patching Theorem.

- By

\[ 0 \rightarrow \Lambda/Q_{r+1} \rightarrow \Lambda/Q_r \oplus \Lambda/P_{r+1} \rightarrow \Lambda/(Q_r, P_{r+1}) \rightarrow 0, \]

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- Set \( \alpha := \lim_{\leftarrow} \alpha_n. \)

- \( \alpha \) is a pseudo-rep. into \( \lim_{\leftarrow} \Lambda/P_n \simeq \Lambda \), since \( \cap_{n=1}^{\infty} P_n = 0. \)

- \( \alpha \) pseudo-rep. into \( \mathcal{K} \) \( \rightarrow \) a rep. \( \rho : G_F \rightarrow \text{GL}_2(\mathcal{K}) \), which has the desired properties.
Proof of the Patching Theorem.

By

\[ 0 \rightarrow \Lambda/Q_{r+1} \rightarrow \Lambda/Q_r \oplus \Lambda/P_{r+1} \rightarrow \Lambda/(Q_r, P_{r+1}) \rightarrow 0, \]

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$$0 \to \Lambda/Q_{r+1} \to \Lambda/Q_r \oplus \Lambda/P_{r+1} \to \Lambda/(Q_r, P_{r+1}) \to 0,$$

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$\alpha$ pseudo-rep. into $\mathcal{K} \rightsquigarrow$ a rep. $\varrho : G_F \to \text{GL}_2(\mathcal{K})$, which has the desired properties.
Wiles’ result and general notations

General strategy of the proof

Tools for the proof

Sketch of the proof

Are there any ordinary primes?
Λ-adic representations attached to Λ-dic forms

Theorem (Wiles)

For a newform $F \in S_{\text{ord}}(\overline{n}, \psi | \Lambda)$, there is a cont. odd irred. rep.

$$\rho_F : G_F \to \text{GL}_2(K)$$

unramified outside $np$ s.t. for every prime $q \nmid np$ we have

$$\text{Tr}(\rho_F)(\text{Frob}_q) = c(q, F)(X) \in \Lambda,$$

$$\det(\rho_F)(\text{Frob}_q) = \psi(q)N(q) \in \Lambda.$$
Theorem (Wiles)

For a newform $F \in S^\text{ord}(\overline{n}, \psi | \Lambda)$, there is a cont. odd irred. rep.

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$$\text{det}(\rho_F)(\text{Frob}_q) = \psi(q)N(q) \in \Lambda.$$

It implies the Main Theorem:

- If $f \in S_k(n, \psi | \mathcal{O})$ ordinary
  $$F \in S^\text{ord}(\overline{n}, \psi \omega^{k-2} | \Lambda)$$
  s.t. $\nu(F) = ef.$

- Schur's Lemma: If $\rho_{f, \lambda} : G_\mathbb{Q} \to \text{GL}_2(L)$ exists, with $L$ a finite extension of $K_{f, \lambda}$, then there is an equivalent rep. $G_\mathbb{Q} \to \text{GL}_2(\mathcal{O}_{f, \lambda}).$

- Ribet: If $\rho_{f, \lambda}$ exists, then it is irreducible.
Theorem (Wiles)

For a newform $F \in S^{\text{ord}}(\overline{n}, \psi \mid \Lambda)$, there is a cont. odd irred. rep.

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- It implies the Main Theorem:

  $$f \in S_k(n, \psi \mid \mathcal{O}) \quad \text{ordinary} \quad \leadsto \quad F \in S^{\text{ord}}(\overline{n}, \psi\omega^{k-2} \mid \Lambda) \quad \text{s.t.} \quad \nu(F) = ef. \quad \leadsto \quad \varrho_F \quad \leadsto \quad \varrho_{f, \lambda} := \nu(\varrho_F)$$

- **Schur's Lemma**: If $\varrho_{f, \lambda} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(L)$ exists, with $L$ a finite extension of $K_{f, \lambda}$, then there is an equivalent rep. $G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathcal{O}_{f, \lambda}).$

- **Ribet**: If $\varrho_{f, \lambda}$ exists, then it is irreducible.
\(\Lambda\)-adic representations attached to \(\Lambda\)-adic forms

**Theorem (Wiles)**

For a newform \(F \in S^{\text{ord}}(\overline{n}, \psi \mid \Lambda)\), there is a cont. odd irred. rep.

\[
\varrho_F: G_F \to \text{GL}_2(\mathcal{K})
\]

unramified outside \(np\) s.t. for every prime \(q \nmid np\) we have

\[
\text{Tr}(\varrho_F)(\text{Frob}_q) = c(q, F)(X) \in \Lambda, \\
\text{det}(\varrho_F)(\text{Frob}_q) = \psi(q)N(q) \in \Lambda.
\]

- It implies the Main Theorem:
  
  \[
  f \in S_k(n, \psi \mid \mathcal{O}) \quad \overset{\text{ordinary}}{\implies} \quad F \in S^{\text{ord}}(\overline{n}, \psi \omega^{k-2} \mid \Lambda) \quad \overset{\text{s.t. } \nu(F) = ef.}{\implies} \quad \varrho_F \implies \varrho_{f, \lambda} := \nu(\varrho_F)
  \]

- **Schur's Lemma**: If \(\varrho_{f, \lambda}: G_\mathbb{Q} \to \text{GL}_2(L)\) exists, with \(L\) a finite extension of \(K_{f, \lambda}\), then there is an equivalent rep. \(G_\mathbb{Q} \to \text{GL}_2(\mathcal{O}_{f, \lambda})\).

- **Ribet**: If \(\varrho_{f, \lambda}\) exists, then it is irreducible.
Warm up: $F = \mathbb{Q}$

- We show:
  1. Eichler-Shimura ($k=2$)
  2. Patching Theorem

\[ \implies \text{Theorem on the previous slide (existence of Galois reps. for } k \geq 1). \]

- For almost all $r \geq 1$

\[
f_r := \nu_{2,r}(F) \in S^\text{ord}_2(Np^r, \psi_{\zeta} | \mathcal{O}[\zeta])
\]

is an eigenform.

- $f_r \xrightarrow{1} \varrho_r: G_{\mathbb{Q}} \to \text{GL}_2(\mathcal{O}[\zeta_r])$ a continuous irreducible odd representation unramified outside $Np$ satisfying that for every $q \nmid Np$

\[
\begin{align*}
\text{Tr}(\varrho_r)(\text{Frob}_q) &= c_q(f_r) = c_q(F)(\chi) \mod{P_{2,r}}, \\
\det(\varrho_r)(\text{Frob}_q) &= \psi(q)q^1 \mod{P_{2,r}}
\end{align*}
\]

where $P_{2,r} = \text{prime of } \Lambda$ associated to $\nu_{2,r}$.

- $\{\varrho_r\}_r \xrightarrow{2} \varrho_F: G_{\mathbb{Q}} \to \text{GL}_2(\mathcal{O})$. 
Warm up: \( F = \mathbb{Q} \)

- We show:
  1. Eichler-Shimura (k=2)
  2. Patching Theorem

\[ \Rightarrow \text{Theorem on the previous slide} \]

(existence of Galois reps. for \( k \geq 1 \)).

- For almost all \( r \geq 1 \)

\[ f_r := \nu_{2,r}(\mathcal{F}) \in S_2^{\text{ord}}(Np^r, \psi \mathcal{O}[\zeta]) \]

is an eigenform.

- \( f_r \overset{1}{\mapsto} \rho_r : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathcal{O}[\zeta]) \) a continuous irreducible odd representation unramified outside \( Np \) satisfying that for every \( q \nmid Np \)

\[ \text{Tr}(\rho_r)(\text{Frob}_q) = c_q(f_r) = c_q(\mathcal{F})(X) \pmod{P_{2,r}}, \]

\[ \det(\rho_r)(\text{Frob}_q) = \psi(q)q^1 \equiv \psi(q)q \pmod{P_{2,r}}, \]

where \( P_{2,r} = \text{prime of } \Lambda \text{ associated to } \nu_{2,r} \).

- \( \{\rho_r\}_r \overset{2}{\mapsto} \rho_{\mathcal{F}} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathcal{K}). \)
Warm up: $F = \mathbb{Q}$

- We show:
  1. Eichler-Shimura ($k=2$)
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  \[ \rightarrow \quad \text{Theorem on the previous slide} \]
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  \text{Tr}(\varrho_r)(\text{Frob}_q) = c_q(f_r) \equiv c_q(\mathcal{F})(X) \pmod{P_{2,r}}, \\
  \det(\varrho_r)(\text{Frob}_q) = \psi(q)q^1 \equiv \psi(q)q \pmod{P_{2,r}},
  \]

  where $P_{2,r} = \text{prime of } \Lambda$ associated to $\nu_{2,r}$.

- $\{\varrho_r\}_r \overset{2}{\mapsto} \varrho_\mathcal{F}: G_\mathbb{Q} \to \text{GL}_2(\mathcal{K})$. 
Warm up: $F = \mathbb{Q}$

- We show:
  1. Eichler-Shimura ($k=2$)
  2. Patching Theorem

  Theorem on the previous slide

  (existence of Galois reps. for $k \geq 1$).

- For almost all $r \geq 1$

  $$f_r := \nu_{2,r}(\mathcal{F}) \in S_2^{\text{ord}}(Np^r, \psi_{\omega_\zeta} | \mathcal{O}[\zeta])$$

  is an eigenform.

- $f_r \underset{1}{\sim} \varphi_r : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathcal{O}[\zeta_r])$ a continuous irreducible odd representation unramified outside $Np$ satisfying that for every $q \nmid Np$

  $$\text{Tr}(\varphi_r)(\text{Frob}_q) = c_q(f_r) \equiv c_q(\mathcal{F})(X) \pmod{P_{2,r}},$$
  $$\det(\varphi_r)(\text{Frob}_q) = \psi(q) q^1 \equiv \psi(q) q \pmod{P_{2,r}},$$

  where $P_{2,r} =$ prime of $\Lambda$ associated to $\nu_{2,r}$.

- $\{\varphi_r\}_r \underset{2}{\sim} \varphi_{\mathcal{F}} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathcal{K})$. 
Warm up: $F = \mathbb{Q}$

- We show:
  1. Eichler-Shimura ($k=2$)
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\[ \begin{align*}
\text{Tr}(\varrho_r)(\text{Frob}_q) &= c_q(f_r) \equiv c_q(F)(X) \pmod{P_{2,r}}, \\
\text{det}(\varrho_r)(\text{Frob}_q) &= \psi(q)q^1 \equiv \psi(q)q \pmod{P_{2,r}},
\end{align*} \]

where $P_{2,r} = \text{prime of } \Lambda \text{ associated to } \nu_{2,r}$.

- $\{\varrho_r\}_r \overset{2}{\sim} \varrho_F : G_{\mathbb{Q}} \to \text{GL}_2(\mathcal{K})$. 
General case

- Assume $d = [F : \mathbb{Q}]$ is even.
- $l \subseteq \mathcal{O}_F$ is a prime s.t $(l, np) = 1$.
- Extending coefficients to $\mathcal{K}$. Set

$$S^{\text{ord}}(\bar{n}, \psi \mid \mathcal{K}) := S^{\text{ord}}(\bar{n}, \psi \mid \Lambda) \otimes_{\Lambda} \mathcal{K}.$$ 

- Space of oldforms with respect to $l$:

$$S^{\text{ord}}(\bar{n}l, \psi \mid \mathcal{K})^{\text{old}} := \{ F(z) + G(lz) \mid F, G \in S^{\text{ord}}(\bar{n}, \psi \mid \mathcal{K}) \}.$$

- Space of newforms with respect to $l$:

$$S^{\text{ord}}(\bar{n}l, \psi \mid \mathcal{K})^{\text{new}} := \mathcal{K} \left\{ F_i(a_{ij}z) \middle| F_i \in S^{\text{ord}}(\bar{m}_i, \psi \mid \Lambda) \text{ newform and } l|\bar{m}_i, (p, a_{ij}) = 1, a_{ij}\bar{m}_i|\bar{n}l \right\}.$$

- Enlarge $\mathcal{K}$ so that it contains the eigenvalues of all eigenforms. Then:

$$S^{\text{ord}}(\bar{n}l, \psi \mid \mathcal{K}) = S^{\text{ord}}(\bar{n}l, \psi \mid \mathcal{K})^{\text{old}} \oplus S^{\text{ord}}(\bar{n}l, \psi \mid \mathcal{K})^{\text{new}}.$$ 

(sum decomposition which does not necessarily hold over $\Lambda$).
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General case

- Assume $d = [F : \mathbb{Q}]$ is even.
- $l \subseteq \mathcal{O}_F$ = a prime s.t $(l, np) = 1$.
- Extending coefficients to $\mathcal{K}$. Set

$$S^{\text{ord}}(\overline{n}, \psi \mid \mathcal{K}) := S^{\text{ord}}(\overline{n}, \psi \mid \Lambda) \otimes_{\Lambda} \mathcal{K}.$$  

- Space of oldforms with respect to $l$:

$$S^{\text{ord}}(\overline{n}l, \psi \mid \mathcal{K})^{\text{old}} := \{ F(z) + G(lz) \mid F, G \in S^{\text{ord}}(\overline{n}, \psi \mid \mathcal{K}) \}.$$  

- Space of newforms with respect to $l$:

$$S^{\text{ord}}(\overline{n}l, \psi \mid \mathcal{K})^{\text{new}} := \mathcal{K} \left\{ F_i(a_{ij}z) \mid \begin{array}{l} F_i \in S^{\text{ord}}(\overline{m}_i, \psi \mid \Lambda) \text{ newform} \\ l|\overline{m}_i, (p, a_{ij}) = 1, a_{ij}\overline{m}_i|\overline{n}l \end{array} \right\},$$

- Enlarge $\mathcal{K}$ so that it contains the eigenvalues of all eigenforms. Then:

$$S^{\text{ord}}(\overline{n}l, \psi \mid \mathcal{K}) = S^{\text{ord}}(\overline{n}l, \psi \mid \mathcal{K})^{\text{old}} \oplus S^{\text{ord}}(\overline{n}l, \psi \mid \mathcal{K})^{\text{new}}.$$  

(sum decomposition which does not necessarily hold over $\Lambda$).
General case

- Assume $d = [F : \mathbb{Q}]$ is even.
- $l \subseteq \mathcal{O}_F$ is a prime s.t $(l, np) = 1$.
- Extending coefficients to $\mathcal{K}$. Set

$$S^{\text{ord}}(\overline{n}, \psi \mid \mathcal{K}) := S^{\text{ord}}(\overline{n}, \psi \mid \Lambda) \otimes_{\Lambda} \mathcal{K}.$$ 

- Space of oldforms with respect to $l$:

$$S^{\text{ord}}(\overline{nl}, \psi \mid \mathcal{K})^{\text{old}} := \{ F(z) + G(lz) \mid F, G \in S^{\text{ord}}(\overline{n}, \psi \mid \mathcal{K}) \}.$$ 

- Space of newforms with respect to $l$:

$$S^{\text{ord}}(\overline{nl}, \psi \mid \mathcal{K})^{\text{new}} := \mathcal{K} \left\{ F_i(a_{ij}z) \mid F_i \in S^{\text{ord}}(\overline{m}_i, \psi \mid \Lambda) \text{ newform and } l \mid \overline{m}_i, (p, a_{ij}) = 1, a_{ij} \overline{m}_i \mid \overline{nl} \right\}.$$ 

- Enlarge $\mathcal{K}$ so that it contains the eigenvalues of all eigenforms. Then:

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\]

- Enlarge \( \mathcal{K} \) so that it contains the eigenvalues of all eigenforms. Then:

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\]

(sum decomposition which does not necessarily hold over \( \Lambda \)).
General case

- Assume $d = [F : \mathbb{Q}]$ is even.
- $\mathfrak{l} \subseteq \mathcal{O}_F$ is a prime s.t. $(\mathfrak{l}, np) = 1$.
- Extending coefficients to $\mathcal{K}$. Set
  
  $$S^{\text{ord}}(\mathfrak{n}, \psi | \mathcal{K}) := S^{\text{ord}}(\mathfrak{n}, \psi | \Lambda) \otimes_\Lambda \mathcal{K}.$$  

- Space of oldforms with respect to $\mathfrak{l}$:
  
  $$S^{\text{ord}}(\mathfrak{n}\mathfrak{l}, \psi | \mathcal{K})^{\text{old}} := \{ F(z) + G(lz) | F, G \in S^{\text{ord}}(\mathfrak{n}, \psi | \mathcal{K}) \}.$$  

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  $$S^{\text{ord}}(\mathfrak{n}\mathfrak{l}, \psi | \mathcal{K})^{\text{new}} := \mathcal{K} \left\{ F_i(a_{ij}z) \left| \begin{array}{l} F_i \in S^{\text{ord}}(\mathfrak{m}_i, \psi | \Lambda) \text{ newform} \\
                            \text{and } l | \mathfrak{m}_i, (p, a_{ij}) = 1, a_{ij} \mathfrak{m}_i | \mathfrak{n}\mathfrak{l} \end{array} \right. \right\},$$  

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(sum decomposition which does not necessarily hold over $\Lambda$).
The congruence module of $\mathcal{F}$

- Set

$$H(\mathcal{F}, l \mid \mathcal{K}) := \{ \mathcal{H} \in S^{\text{ord}}(\mathfrak{n}l, \psi \mid \mathcal{K})^{\text{new}} \mid \mathcal{H} = \mathcal{G} - u \mathcal{F} - v \mathcal{F}(lz),$$

with $\mathcal{G} \in S^{\text{ord}}(\mathfrak{n}l, \psi \mid \Lambda)$, $u, v \in \mathcal{K}\}.$

The congruence module for $\mathcal{F}$ is

$$C(\mathcal{F}, l \mid \mathcal{K}) := H(\mathcal{F}, l \mid \mathcal{K})/ S^{\text{ord}}(\mathfrak{n}l, \psi \mid \mathcal{K})^{\text{new}} \cap S^{\text{ord}}(\mathfrak{n}l, \psi \mid \Lambda).$$

It measures how far the direct sum decomposition over $\mathcal{K}$ fails to be a direct sum decomposition over $\Lambda$.

- Let $\mathbb{T} \subseteq \text{End}(S^{\text{ord}}(\mathfrak{n}l, \psi \mid \mathcal{K})^{\text{new}})$ denote the ring generated over $\Lambda$ by the Hecke operators $\mathcal{T}(m)$ for $(m, l) = 1$.

- Set

$$l_{\mathcal{F}} = \text{Ann}(C(\mathcal{F}, l \mid \mathcal{K})) \subseteq \mathbb{T}.$$

$$\mathcal{T}(m) - c(m, \mathcal{F})(X) \in l_{\mathcal{F}} \quad \text{for} \ (m, l) = 1 \quad \Rightarrow \quad \mathbb{T}/l_{\mathcal{F}} \cong \Lambda/b_{\mathcal{F}, l}$$

for some ideal $b_{\mathcal{F}, l} \subseteq \Lambda$. 
The congruence module of $\mathcal{F}$

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$$\mathcal{T}(m) - c(m, \mathcal{F})(X) \in l_{\mathcal{F}} \quad \text{for } (m, l) = 1 \quad \Rightarrow \quad \mathbb{T} / l_{\mathcal{F}} \simeq \Lambda / b_{\mathcal{F}, l}$$

for some ideal $b_{\mathcal{F}, l} \subseteq \Lambda$. 
The congruence module of $\mathcal{F}$

- Set
\[
H(\mathcal{F}, l \mid \mathcal{K}) := \{ H \in S^{\text{ord}}(\overline{m}, \psi \mid \mathcal{K})^{\text{new}} \mid H = G - u \mathcal{F} - v \mathcal{F}(lz), \]
with $G \in S^{\text{ord}}(\overline{m}, \psi \mid \Lambda)$, $u, v \in \mathcal{K}\}.
\]

The congruence module for $\mathcal{F}$ is
\[
C(\mathcal{F}, l \mid \mathcal{K}) := H(\mathcal{F}, l \mid \mathcal{K}) / S^{\text{ord}}(\overline{m}, \psi \mid \mathcal{K})^{\text{new}} \cap S^{\text{ord}}(\overline{m}, \psi \mid \Lambda).
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It measures how far the direct sum decomposition over $\mathcal{K}$ fails to be a direct sum decomposition over $\Lambda$.

- Let $\mathbb{T} \subseteq \text{End}(S^{\text{ord}}(\overline{m}, \psi \mid \mathcal{K})^{\text{new}})$ denote the ring generated over $\Lambda$ by the Hecke operators $\mathcal{T}(m)$ for $(m, l) = 1$.

- Set
\[
l_{\mathcal{F}} = \text{Ann}(C(\mathcal{F}, l \mid \mathcal{K})) \subseteq \mathbb{T}.
\]

\[
\mathcal{T}(m) - c(m, \mathcal{F})(X) \in l_{\mathcal{F}} \quad \text{for} \quad (m, l) = 1 \quad \Rightarrow \quad \mathbb{T}/l_{\mathcal{F}} \cong \Lambda/b_{\mathcal{F}, l} \quad \text{for some ideal} \ b_{\mathcal{F}, l} \subseteq \Lambda.
\]
The congruence module of $\mathcal{F}$

- Set
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  H(\mathcal{F}, l \mid \mathcal{K}) := \{ \mathcal{H} \in S^{\text{ord}}(\overline{m}, \psi \mid \mathcal{K})^{\text{new}} \mid \mathcal{H} = \mathcal{G} - u \mathcal{F} - v \mathcal{F}(lz), \\
  \text{with } \mathcal{G} \in S^{\text{ord}}(\overline{m}, \psi \mid \Lambda), \ u, v \in \mathcal{K} \}.
  \]

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- Set
  \[
  l_{\mathcal{F}} = \text{Ann}(C(\mathcal{F}, l \mid \mathcal{K})) \subseteq \mathbb{T}.
  \]

  \[
  \mathcal{T}(m) - c(m, \mathcal{F})(X) \in l_{\mathcal{F}} \quad \Rightarrow \quad \mathbb{T} / l_{\mathcal{F}} \cong \Lambda / b_{\mathcal{F}, l}
  \]

  for some ideal $b_{\mathcal{F}, l} \subseteq \Lambda$. 

Francesc Fité (Universität Duisburg-Essen)
There are infinitely many specializations of each $\mathcal{F}_i(a_{ij}z)$ in the special basis satisfying hypothesis ii) of Carayol’s Theorem at $l \not\mid l$. Consider the rep.

$$\bigoplus \varrho \mathcal{F}_i(a_{ij}z) \otimes \mathcal{K} : G_F \to \text{GL}_2(A)$$

Endow $A$ with an action of $T$ by transport of structure. The map

$T \otimes \mathcal{K} \to A$ induced by $\mathcal{T}(q) \mapsto \prod_{i,j} c(q, \mathcal{F}_i(a_{ij}z))$

is an isomorphism of $T \otimes \mathcal{K}$-modules.

We obtain an odd rep.

$$\varrho : G_F \to \text{GL}_2(T \otimes \mathcal{K})$$

s.t. for $q \nmid n \mid p$

$$\text{Tr}(\varrho)(\text{Frob}_q) = \prod_{i,j} c(q, \mathcal{F}_i(a_{ij}z)) = \mathcal{T}(q) \in T.$$
There are infinitely many specializations of each $\mathcal{F}_i(a_{i,j}z)$ in the special basis satisfying hypothesis ii) of Carayol’s Theorem at $l$.

Set $A := \prod_{i,j} \mathcal{K}$. Consider the rep.

$$\bigoplus \varrho_{\mathcal{F}_i(a_{ij}z)} \otimes \mathcal{K}: G_F \to \text{GL}_2(A)$$

Endow $A$ with an action of $T$ by transport of structure. The map

$$T \otimes \mathcal{K} \to A \quad \text{induced by} \quad T(q) \mapsto \prod_{i,j} c(q, \mathcal{F}_i(a_{ij}z))$$

is an isomorphism of $T \otimes \mathcal{K}$-modules.

We obtain an odd rep.

$$\varrho: G_F \to \text{GL}_2(T \otimes \mathcal{K})$$

s.t. for $q \nmid np$

$$\text{Tr}(\varrho)(\text{Frob}_q) = \prod_{i,j} c(q, \mathcal{F}_i(a_{ij}z)) = T(q) \in T.$$
There are infinitely many specializations of each $\mathcal{F}_i(a_{i,j}z)$ in the special basis satisfying hypothesis $ii)$ of Carayol’s Theorem at $l 
mid \varpi$.

Set $A := \prod_{i,j} \mathcal{K}$. Consider the rep.

$$\bigoplus \varrho \mathcal{F}_i(a_{i,j}z) \otimes \mathcal{K} : G_F \to \text{GL}_2(A)$$

Endow $A$ with an action of $\mathbb{T}$ by transport of structure. The map

$$\mathbb{T} \otimes \mathcal{K} \to A$$

induced by

$$\mathcal{T}(q) \mapsto \prod_{i,j} c(q, \mathcal{F}_i(a_{i,j}z))$$

is an isomorphism of $\mathbb{T} \otimes \mathcal{K}$-modules.

We obtain an odd rep.

$$\rho : G_F \to \text{GL}_2(\mathbb{T} \otimes \mathcal{K})$$

s.t. for $q \nmid n \nmid \rho$

$$\text{Tr} (\rho)(\text{Frob}_q) = \prod_{i,j} c(q, \mathcal{F}_i(a_{i,j}z)) = \mathcal{T}(q) \in \mathbb{T}.$$
There are infinitely many specializations of each $\mathcal{F}_i(a_{i,j}z)$ in the special basis satisfying hypothesis ii) of Carayol’s Theorem at $l \sim \mathcal{O}_F(a_{i,j}z)$.

Set $A := \prod_{i,j} \mathcal{K}$. Consider the rep.

$$\bigoplus \mathcal{O}_F(a_{i,j}z) \otimes \mathcal{K} : G_F \to \text{GL}_2(A)$$

Endow $A$ with an action of $\mathbb{T}$ by transport of structure. The map

$$\mathbb{T} \otimes \mathcal{K} \to A \quad \text{induced by} \quad \mathcal{T}(q) \mapsto \prod_{i,j} c(q, \mathcal{F}_i(a_{i,j}z))$$

is an isomorphism of $\mathbb{T} \otimes \mathcal{K}$-modules.

We obtain an odd rep.

$$\varrho : G_F \to \text{GL}_2(\mathbb{T} \otimes \mathcal{K})$$

s.t. for $q \nmid nlp$

$$\text{Tr}(\varrho)(\text{Frob}_q) = \prod_{i,j} c(q, \mathcal{F}_i(a_{i,j}z)) = \mathcal{T}(q) \in \mathbb{T}.$$
Let $\pi$ be the odd pseudo rep. associated to $\varrho$.

$\text{Tr}(\varrho) \in \mathbb{T} \Rightarrow \text{Tr}(\pi) \in \mathbb{T} \Rightarrow \pi$ is takes values in $\mathbb{T}$.

Let $\overline{\pi}$ be the pseudo-rep. $\pi \pmod{l_\mathcal{F}}$:

$$\text{Tr}(\overline{\pi})(\text{Frob}_q) = T(q) = c(q, F)(X) \in \mathbb{T}/l_\mathcal{F} = \Lambda/b_{\mathcal{F}}$$

Take a prime $b_{\mathcal{F}, l} \subseteq Q \subseteq \Lambda$.

Let $\pi_Q$ be the pseudo-rep. $\overline{\pi} \pmod{Q}$.

$\pi_Q \leftrightarrow \varrho_Q$.

The proof continues with a difficult and technical argument to ensure that, by making distinct choices of $l$, we may find infinitely many distinct primes $b_{\mathcal{F}, l} \subseteq Q \subseteq \Lambda$.

One concludes by patching together the $\varrho_Q$'s.
Let $\pi$ be the odd pseudo rep. associated to $\varrho$.

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$$\text{Tr}(\overline{\pi})(\text{Frob}_q) = \mathcal{T}(q) = c(q, \mathcal{F})(X) \in \mathbb{T}/l_F \cong \Lambda/b_{\mathcal{F},l}.$$

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Let \( \overline{\pi} \) be the pseudo-rep. \( \pi \mod l_\mathcal{F} \):

\[
\text{Tr}(\overline{\pi})(\text{Frob}_q) = \mathcal{T}(q) \equiv c(q, \mathcal{F})(X) \in \mathbb{T}/l_\mathcal{F} \cong \Lambda/b_{\mathcal{F},l}
\]

- Take a prime \( b_{\mathcal{F},l} \subseteq Q \subseteq \Lambda \).
- Let \( \pi_Q \) be the pseudo-rep. \( \pi \mod Q \).
- \( \pi_Q \rightsquigarrow \varrho_Q \).
- The proof continues with a difficult and technical argument to ensure that, by making distinct choices of \( l \), we may find infinitely many distinct primes \( b_{\mathcal{F},l} \subseteq Q \subseteq \Lambda \).
- One concludes by patching together the \( \varrho_Q \)'s.
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Let $\bar{\pi}$ be the pseudo-rep. $\pi \pmod{l_{\mathcal{F}}}$:

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Take a prime $b_{\mathcal{F},l} \subseteq Q \subseteq \Lambda$.

Let $\pi_Q$ be the pseudo-rep. $\bar{\pi} \pmod{Q}$.

$\pi_Q \rightsquigarrow \varrho_Q$.

The proof continues with a difficult and technical argument to ensure that, by making distinct choices of $l$, we may find infinitely many distinct primes $b_{\mathcal{F},l} \subseteq Q \subseteq \Lambda$.

One concludes by patching together the $\varrho_Q$'s.
Let $\pi$ be the odd pseudo rep. associated to $\varrho$.

$\Tr(\varrho) \in T \Rightarrow \Tr(\pi) \in T \Rightarrow \pi$ is takes values in $T$.

Let $\overline{\pi}$ be the pseudo-rep. $\pi \pmod{l_{\mathcal{F}}}$:

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Take a prime $b_{\mathcal{F}, l} \subseteq Q \subseteq \Lambda$.

Let $\pi_Q$ be the pseudo-rep. $\overline{\pi} \pmod{Q}$.

$\pi_Q \mapsto \varrho_Q$.

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Take a prime $b_{F,1} \subseteq Q \subseteq \Lambda$.

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The proof continues with a difficult and technical argument to ensure that, by making distinct choices of $l$, we may find infinitely many distinct primes $b_{F,1} \subseteq Q \subseteq \Lambda$.

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Take a prime $b_{\mathcal{F},l} \subseteq Q \subseteq \Lambda$.

Let $\pi_Q$ be the pseudo-rep. $\overline{\pi} \pmod{Q}$.

$\pi_Q \longmapsto \varrho_Q$.

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One concludes by patching together the $\varrho_Q$'s.
• Let $\pi$ be the odd pseudo rep. associated to $\varrho$.

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• Take a prime $b_{F,l} \subseteq Q \subseteq \Lambda$.

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• $\pi_Q \quad \quad \varrho Q$.

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• One concludes by patching together the $\varrho Q$'s.
Wiles’ result and general notations

General strategy of the proof

Tools for the proof

Sketch of the proof

Are there any ordinary primes?
On the existence of ordinary primes

- For simplicity, take $F = \mathbb{Q}$.
- For $f(q) = \sum_{n \geq 1} c_n q^n \in S_k(N, \psi)$, set

$$\Sigma := \{ p \text{ prime} \mid c_p \not\equiv 0 \pmod{p} \}.$$

- For general $f$, is it known whether $\Sigma$:
  - has a positive density?
  - contains infinitely many primes?
  - is at least non empty?
- If $f$ has CM: an affirmative answer is well-known.
  So assume, from now on, that $f$ does not have CM.
- For $k > 3$: open.
- For $k \leq 3$: $\Sigma$ has a positive density.

Theorem (Serre ’81)

The set $S = \{ p \text{ prime} \mid c_p = 0 \}$ has zero density.
On the existence of ordinary primes

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- For simplicity, take $F = \mathbb{Q}$.
- For $f(q) = \sum_{n \geq 1} c_n q^n \in S_k(N, \psi)$, set
  $$\Sigma := \{ p \text{ prime} \mid c_p \not\equiv 0 \pmod{p} \}.$$  
- For general $f$, is it known whether $\Sigma$:
  - has a positive density?
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- If $f$ has CM: an affirmative answer is well-known.
  So assume, from now on, that $f$ does not have CM.
- For $k > 3$: open.
- For $k \leq 3$: $\Sigma$ has a positive density.

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**Theorem (Serre ’81)**

*The set $S = \{ p \text{ prime \mid } c_p = 0 \}$ has zero density.*
On the existence of ordinary primes

- Suppose that $k \leq 2$. Then, the Ramanujan-Petersson inequality

$$|c_p| \leq 2p^{(k-1)/2} \leq 2\sqrt{p}$$

implies that almost all primes not in $\Sigma$ are in $S$. Serre’s theorem $\Rightarrow \Sigma$ has density 1.

- Suppose now that $k = 3$.
  - $\varrho_f$ is odd $\Rightarrow \text{Tr}(\varrho_f)(\sigma) = 0$.
  - Pick a prime $\ell > 2$. Then, the set

$$S_\ell := \{ p \text{ prime} \mid c_p \equiv 0 \pmod{\ell} \}$$

has a positive density.
  - The Ramanujan-Petersson inequality

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implies now that every prime in $S_\ell$ and not in $\Sigma$ is in $S$.
  - Serre’s Theorem $\Rightarrow$ the density of $\Sigma$ is at least the density of $S_\ell$. 
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