Freeness of local Hecke components of the ℓ-adic cohomology of Shimura curves

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The modular case: minimal level

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- Let $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O})$ be the ℓ-adic Galois representation associated to f and let $\overline{\rho}$ be its reduction modulo ℓ.
- Assumption about $\bar{\rho}$:

$$\overline{\rho} \text{ is absolutely irreducible;} \tag{1}$$

$$\text{if } p|N \text{ then } \overline{\rho}(I_p) \neq 1; \tag{2}$$

$$\text{if } \ell = 3,\ \overline{\rho} \text{ is not induced from a character of } \mathbf{Q}(\sqrt{-3}). \tag{3}$$

- Consider deformations of $\bar{\rho}$ satisfying the following conditions:
  a) ρ is unramified outside N.
  b) if $p|N$ then $\rho(I_p) \simeq \overline{\rho}(I_p)$;
  c) ρ is semistable at ℓ
  d) $\det(\rho) = \epsilon$, where $\epsilon : G_{\mathbf{Q}} \to \mathbb{Z}_\ell^\times$ is the cyclotomic character.

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- This is a deformation condition. Let $\mathcal{R}$ be the universal deformation ring which parametrizes such representations.
- Let $\mathcal{H}(N)$ be the Hecke algebra acting on $S_2(\Gamma_0(N))$; then f defines a maximal ideal $\mathfrak{m}$ in $\mathcal{H}(\mathcal{N}) \otimes_{\mathbb{Z}} \mathcal{O}$. Define $\mathbf{T} = (\mathcal{H}(N) \otimes_{\mathbb{Z}} \mathcal{O})_{\mathfrak{m}}$, $M = H^1(\Gamma_0(N), \mathcal{O})_{\mathfrak{m}}$. There is a natural map $\mathcal{R} \to T$.

Theorem (Taylor, Wiles)

- *$\mathcal{R} \simeq T$ is an isomorphism of complete intersection rings*
- *M is free of rank 2 over T*

The modular case: adding primes to the level

- Let S be a set of primes disjoint from $N\ell$ and consider deformations ρ of $\overline{\rho}$ allowing ramification at primes in S, that is
  a) ρ is unramified outside NS.
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- This is a deformation condition. Let $\mathcal{R}_S$ be the universal deformation ring which parametrizes these deformations.
- One can define a suitable level N_S, a maximal ideal $\mathfrak{m}_S$ of $\mathcal{H}(N_S) \otimes_{\mathbb{Z}} \mathcal{O}$ associated to f and define $T_S = (\mathcal{H}_S \otimes_{\mathbb{Z}} \mathcal{O})_{\mathfrak{m}_S}$, $M_S = H^1(\Gamma_0(N_S), \mathcal{O})_{\mathfrak{m}_S}$. There is a natural map $\mathcal{R}_S \to T_S$.

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A generalization using Shimura curves

- Let now $f \in S(\Gamma_0(N\Delta'\ell^2))$ be a modular newform of weight 2,
  - supercuspidal of type $\tau = \chi \oplus \chi^\sigma$ at ℓ, where χ is a non-trivial character of $\mathbb{F}_{\ell^2}^\times$, trivial over $\mathbb{F}_\ell^\times$.
  - special at all primes dividing $N\Delta'$.
- Let ρ be the Galois representation associated to f and let $\overline{\rho}$ be its reduction modulo ℓ;
- we impose the following conditions on $\overline{\rho}$:

$$\overline{\rho} \text{ is absolutely irreducible;} \tag{4}$$

$$\text{if } p|\Delta' N \text{ then } \overline{\rho}(I_p) \neq 1; \tag{5}$$

$$\mathrm{End}_{\overline{\mathbf{F}}_\ell[G_\ell]}(\overline{\rho}_\ell) = \overline{\mathbf{F}}_\ell. \tag{6}$$

$$\text{if } \ell = 3,\ \overline{\rho} \text{ is not induced from a character of } \mathbf{Q}(\sqrt{-3}). \tag{7}$$

The deformation problem: minimal case

- Consider deformation ρ of $\bar{\rho}$ satisfying
  a) ρ is unramified outside $N\Delta'\ell$.
  b) if $p|\Delta' N$ then $\rho(I_p) \simeq \overline{\rho}(I_p)$;
  c) ρ_ℓ is weakly of type τ (see Conrad Diamond Taylor 1999)
  d) $\det(\rho) = \epsilon$, where $\epsilon : G_{\mathbf{Q}} \to \mathbb{Z}_\ell^\times$ is the cyclotomic character.
- This is a deformation condition. Let $\mathcal{R}$ be the universal deformation ring which parametrizes representations of this kind.

- Suppose that Δ' is the product of an odd number of primes and put $\Delta = \Delta'\ell$
- Let $R(N)$ be an Eichler order in the quaternion algebra of discriminant Δ, and consider

$$\Phi_0(N) = R(N)^{(1)}$$

- There an action of $\mathbb{F}_{\ell^2}^\times$ over $H^1(\Phi_0(N), \mathcal{O})$, compatible with the Hecke action.
- Let $\mathcal{H}(N)^\chi$ be the Hecke algebra acting on $H^1(\Phi_0(N), \mathcal{O})^\chi$; then f defines a maximal ideal $\mathfrak{m}$ in $\mathcal{H}(N)^\chi \otimes_{\mathbb{Z}} \mathcal{O}$. Define $\mathbf{T} = (\mathcal{H}(N)^\chi \otimes_{\mathbb{Z}} \mathcal{O})_{\mathfrak{m}}$, $M = H^1(\Gamma_0(N), \mathcal{O})^\chi_{\mathfrak{m}}$. There is a natural map $\mathcal{R} \to T$.

Using a suitable Taylor-Wiles system one can prove:

Theorem (generalized by M. Ciavarella for the case with nebentypus)

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Adding primes in the level?

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The techniques used in the modular case for adding primes in the level have been axiomatized by Diamond (Inventiones 1997)

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- Let $\mathfrak{p}$ be a prime ideal of R in the support of M. Let I be the annihilator of $\mathfrak{p}$ in R.

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- Let $\mathfrak{p}$ be a prime ideal of R in the support of M. Let I be the annihilator of $\mathfrak{p}$ in R.
- Define

$$\Omega(M) = \frac{M}{M[\mathfrak{p}] + M[I]}.$$

Theorem (Diamond 1997)

The following statements are equivalent:

i) $rk_{\mathcal{O}}(M) \leq d \cdot rk_{\mathcal{O}}(T)$ and $length_{\mathcal{O}} \Omega \geq d \cdot lenght_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2)$;

ii) *R is a complete intersection and M is free (of rank d) over R.*

Application of Diamond's theorem to add primes in the level

- Fix complete local noetherian $\mathcal{O}$-algebras R, R' with $\mathcal{O}$-algebra homomorphisms $\varphi : R' \to R$, $\pi : R \to \mathcal{O}$; define $\pi' = \pi \circ \varphi$.

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- Suppose that there exists an injective R'-homomorphism $\alpha : M \to M'$ with cokernel torsion free.
- Define $T = R/Ann_R(M)$, $T' = R'/Ann'_R(M')$; then φ induces a surjective map $T' \to T$.

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- Suppose that there exists an injective R'-homomorphism $\alpha : M \to M'$ with cokernel torsion free.
- Define $T = R/Ann_R(M)$, $T' = R'/Ann'_R(M')$; then φ induces a surjective map $T' \to T$.
- Suppose that $\mathfrak{p} = \mathrm{Ker}(\pi)$ is in the support of M and let $\mathfrak{p}' = \varphi^{-1}(\mathfrak{p})$.
- Then α induces an isomorphism $M[\mathfrak{p}] \simeq M'[\mathfrak{p}']$, which are both free $\mathcal{O}$-modules of finite rank d.

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- Suppose moreover that

$$rk_{\mathcal{O}}(\Omega') - rk_{\mathcal{O}}(\Omega) \geq d \cdot (lenght_{\mathcal{O}}(\mathfrak{p}'/\mathfrak{p}'^2) - lenght_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2)) \quad (8)$$

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- Then by Diamond theorem we can conclude that R' is complete intersection and M' is free over R'.

- Suppose moreover that M (resp. M') is equipped with a alternating, T (resp. T') bi-linear, $\mathcal{O}$-perfect pairing $< \cdot, \cdot >$ (resp. $< \cdot, \cdot >'$) and let $\beta : M' \to M$ be the transpose of α w.r.t. these pairings.

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- It is easy to see that Ω (resp. Ω') is isomorphic to the cokernel of the map $M[\mathfrak{p}] \to \mathrm{Hom}(M[\mathfrak{p}], \mathcal{O})$ (resp. $M'[\mathfrak{p}'] \to \mathrm{Hom}(M'[\mathfrak{p}'], \mathcal{O})$), so that Ω and Ω' have finite lenght over $\mathcal{O}$.

- If x, y is a $\mathcal{O}$-basis for $M[\mathfrak{p}]$ it is easy to see that $rk_{\mathcal{O}}(\Omega) = d \cdot v_\lambda(< x, y >)$, and the same holds for Ω' and any basis x', y' of $M'[\mathfrak{p}']$.

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$$lenght_{\mathcal{O}}(\mathfrak{p}'/\mathfrak{p}'^2) - lenght_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2) = lenght_{\mathcal{O}}(K),$$

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$$v_\lambda(c) \geq lenght_{\mathcal{O}}(K). \qquad (9)$$

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What we can do...

- Starting from cup product, it is easy to define the pairing.

Definition of the map α

- There is an element $\eta_p \in B$, with reduced norm p such that $\eta_p \Phi_0(Np)\eta_p^{-1} \subseteq \Phi_0(N)$;
- Inclusion and conjugation by η_p induce two maps $\alpha_1, \alpha_2 : H^1(\Phi_0(N), \mathcal{O}) \to H^1(\Phi_0(Np), \mathcal{O})$, equivariant for all Hecke operators, except T_p
- There is a suitable linear combination α of α_1, α_2 mapping M in M'.
- Since $\mathfrak{m}$ is non-Eisenstein, α is injective.
- We can compute $c = \beta \circ \alpha$
- Using local deformation theory we can show then $c \geq lenght_{\mathcal{O}}(K)$

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What we miss...

- $coker(\alpha)$ is torsion free

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Proposition

$$\Phi = \left(R(N) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{p}\right]\right)^{(1)}.$$

- If the amalgama Φ had the subgroup congruence property (i.e every subgroup of finite index is a congruence subgroup), then the same approach used in the modular case would be applicable to our Shimura case:

Proposition

Assume that Φ has the subgroup congruence property. Then coker(α) is torsion free.

- the proof considers the Lyndon exact sequence

$$...H^1(\Phi, k)^\chi \longrightarrow (H^1(\Phi_0(N), k)^\chi)^2 \xrightarrow{\alpha\otimes k} H^1(\Phi_0(Nq), k)^\chi...$$

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Assume that Φ has the subgroup congruence property. Then coker(α) is torsion free.

- the proof considers the Lyndon exact sequence

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- Unfortunately the congruence subgroup property for p-arithmetic subgroups of $B^{\times}$ is conjectured, but not known. For a reference about this problem see Rapinchuk (1999).

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- Diamond and Taylor (1994) prove $coker(\alpha)$ torsion free in Shimura cases when $\mathfrak{m}$ is non-Eisenstein, and $\ell \nmid \Delta$.