MacLane valuations

(after Guàrdia, Montes, Nart, Vaquié)

STNB 2014

Discrete valuation on a field K

Fix a **valuation** $v : K \longrightarrow \mathbb{Z} \cup \{\infty\}$

Only two properties in the toolbox

- $v : K^* \longrightarrow \mathbb{Z}$ is a group homomorphism
- $v(a+b) \geq \min\{v(a), v(b)\}$ (with $=$ if $v(a) \neq v(b)$)

Valuation ring (principal local ring)

$$\mathcal{O} := \{a \in K \mid v(a) \geq 0\}$$

Maximal ideal

$$\mathfrak{m} := \{a \in K \mid v(a) > 0\} = \pi\,\mathcal{O}$$

Residue field

$$\mathbb{F} := \mathcal{O}/\mathfrak{m}$$

Extensions to the function field $K(x)$

$$\mathbb{V} := \{\, \mu : K(x)^* \longrightarrow \mathbb{Q} \ \ \textbf{discrete} \ \mid \ \mu_{|K} = v, \ \ \mu(x) \geq 0 \,\}$$

Discrete extensions

The value group $\Gamma(\mu) := \mu\left(K(x)^*\right)$ is **cyclic**

$$\begin{array}{ccc} K(x)^* & \xrightarrow{\ \mu\ } & \Gamma(\mu) = \dfrac{\mathbb{Z}}{e(\mu)} \\ \Big| & & \Big| \\ K^* & \xrightarrow{\ v\ } & \mathbb{Z} \end{array}$$

$e(\mu) \in \mathbb{N}$ is the **ramification index** of μ

How to handle $\mathbb{V}$?

Graded algebra of a valuation μ on $K(x)$

Equivalence relation in $K[x]$ with respect to μ

$$h \sim_\mu g \iff g = h = 0 \quad \text{or} \quad \boxed{\mu(h-g) > \mu(g)}$$

μ-equivalence class of $g \in K[x]$

$$H_\mu(0) := 0 \qquad H_\mu(g) := \boxed{g + \mathcal{P}^+_{\mu(g)}} \quad \text{for } g \neq 0$$

$$\mathcal{P}^+_\alpha := \{g \in K[x] \mid \mu(g) > \alpha\} \subset \mathcal{P}_\alpha := \{g \in K[x] \mid \mu(g) \geq \alpha\}$$

$$H_\mu : K[x] \longrightarrow \mathcal{G}r(\mu) := \bigoplus_{\alpha \in \Gamma(\mu)} \mathcal{P}_\alpha / \mathcal{P}^+_\alpha$$

Graded ring structure

$$H_\mu(gh) = H_\mu(g)\, H_\mu(h) \quad \text{but} \quad H_\mu(g+h) \neq H_\mu(g) + H_\mu(h)$$

Graded algebra of a valuation μ on $K(x)$

Distinguished classes in $K[x]$

Those with μ-value **zero**

Subring $\Delta(\mu)$

$$\begin{array}{ccccccc}
\mathfrak{m} & \hookrightarrow & \mathcal{O} & \twoheadrightarrow & \mathbb{F} & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{P}_0^+ & \hookrightarrow & \mathcal{P}_0 & \twoheadrightarrow & \Delta(\mu) & \hookrightarrow & \mathcal{G}r(\mu)
\end{array}$$

The **integral domain** $\mathcal{G}r(\mu)$ is an $\mathbb{F}$**-algebra**

Graded algebra of a valuation μ on $K(x)$

$$H_\mu \,:\, K[x] \longrightarrow \mathcal{G}r(\mu)$$

Divisibility relation in $K[x]$ with respect to μ

$$h \mid_\mu g \iff H_\mu(h) \text{ **divides** } H_\mu(g) \text{ in } \mathcal{G}r(\mu)$$

$$h \mid_\mu g \iff \boxed{g \sim_\mu h\, f} \text{ with } f \in K[x]$$

Minimality in $K[x]$ with respect to μ

A polynomial $\phi \in K[x]$ is μ-**minimal** if

$$\deg\phi = \min\left\{\deg g \;\middle|\; \phi \mid_\mu g \neq 0\right\}$$

Fundamental example: the valuation μ_0

$$g = \sum_{s\geq 0} a_s\, x^s \in K[x]$$

$$\boxed{\mu_0(g) := \min_{s\geq 0}\ \mu_0(a_s\, x^s)}$$

$$\boxed{\mu_0(a_s) := v(a_s)} \qquad \boxed{\mu_0(x) := 0}$$

The *smallest* valuation in $\mathbb{V}$

$\boxed{\mu_0(g) \leq \mu(g)}$ for all $g \in K[x]$ and all $\mu \in \mathbb{V}$

$$\Delta(\mu_0) = \mathcal{O}[x]/\mathfrak{m}[x] \ \simeq\ \mathbb{F}[y]$$

$$H_{\mu_0}(g) \longmapsto \overline{g}$$

$$\Gamma(\mu_0) = \mathbb{Z}$$

$$\mathcal{G}r(\mu_0) \simeq \bigoplus_{\alpha\in\mathbb{Z}} H_{\mu_0}(\pi)^{\alpha}\ \mathbb{F}[y]$$

The simplest example

$\phi = x$ is μ_0-**minimal**

Another definition of μ-minimality

For a fixed polynomial $\phi \in K[x] \setminus K$, let

$$g = \sum_{s \geq 0} g_s \, \phi^s \qquad (\deg g_s < \deg \phi)$$

be the **canonical** ϕ**-expansion** of a polynomial $g \in K[x]$

For a valuation μ on $K(x)$,

ϕ is μ-**minimal**

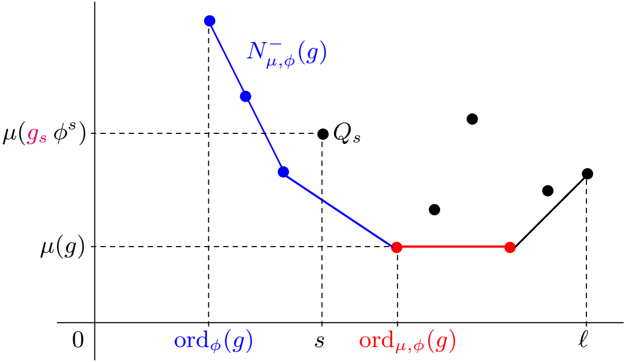
$\Updownarrow$

for all $g \in K[x]$, $\boxed{\mu(g) = \min_{s \geq 0} \mu(g_s \, \phi^s)}$

Newton polygon operator attached to ϕ μ-minimal

$$g = g_0 + g_1\,\phi + \cdots + g_\ell\,\phi^\ell \in K[x] \qquad (\deg g_s < \deg \phi, \quad g_\ell \neq 0)$$

$$\boxed{N_{\mu,\phi}(g)} := \text{lower convex hull of } \{Q_s \mid 0 \le s \le \ell\}$$



$$\mathrm{ord}_{\mu,\phi}(g) = \max\{\, s \ge 0 \mid \phi^s \mid_\mu g \,\}$$

Augmentations of a valuation μ on $K(x)$

Key polynomials for μ

$KP(\mu) := \{\, \phi \in K[x]$ monic, μ-**minimal**, μ-**irreducible** $\}$

ϕ is μ-**irreducible** $\iff$ $H_\mu(\phi)$ is **prime** in $\mathcal{G}r(\mu)$

Example

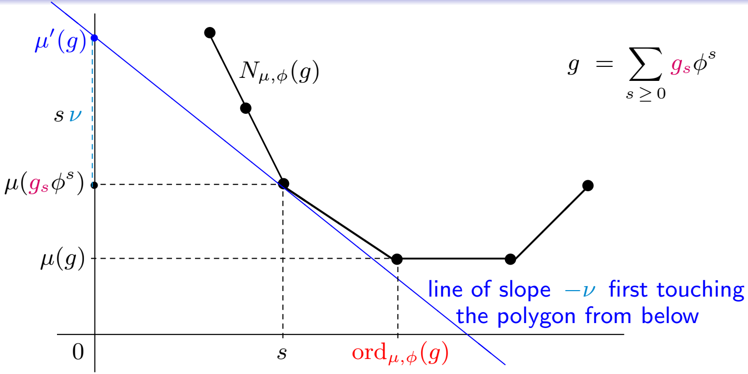
$KP(\mu_0) = \{\, \phi \in \mathcal{O}[x]$ monic with $\overline{\phi}$ irreducible in $\mathbb{F}[y]\,\}$

Key definition

Given $\phi \in KP(\mu)$ and $\nu > 0$, the **augmentation** $\boxed{\mu \xrightarrow{\phi,\,\nu} \mu'}$ is

$$\mu'\Big(\sum_{s\ge 0} g_s\,\phi^s\Big) := \min_{s\ge 0}\{\,\mu(g_s\,\phi^s) + s\,\nu\,\}$$

Augmentations of a valuation μ on $K(x)$



$$\boxed{\mu'(g) = \mu(g)} \iff \mathrm{ord}_{\mu,\,\phi}(g) = 0 \iff \boxed{\mu(g) = \mu(g_0)}$$

$$\Updownarrow$$

$$\boxed{\phi \nmid_\mu g}$$

Augmentations of a valuation μ on $K(x)$

Is μ' as expected?

μ' is a **valuation** on $K(x)$ with $\boxed{\mu \le \mu'}$ and $\phi \in KP(\mu')$

For which polynomials μ' grows strictly?

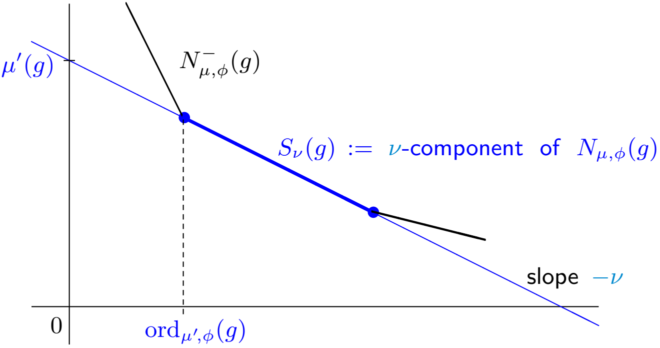
The **kernel** of the graded algebra homomorphism

$$\begin{array}{rcl} \mathcal{G}r(\mu) & \longrightarrow & \mathcal{G}r(\mu') \\ H_\mu(g) & \longmapsto & g + \mathcal{P}^+_{\mu(g)}(\mu') = \begin{cases} H_{\mu'}(g) & \text{if} \quad \mu'(g) = \mu(g) \\ 0 & \text{if} \quad \mu'(g) > \mu(g) \end{cases} \end{array}$$

is (the **prime** ideal)

$$\boxed{H_\mu(\phi)\,\mathcal{G}r(\mu)}$$

Augmentations of a valuation μ on $K(x)$



Theorem of the product – pocket version

For $g, h \in K[x]$, $g, h \neq 0$,

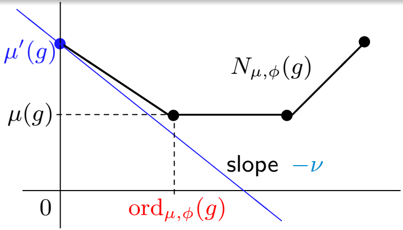
$$S_\nu(gh) = S_\nu(g) + S_\nu(h)$$

Augmentations of a valuation μ on $K(x)$

Units in $\mathcal{G}r(\mu')$

For $g \in K[x]$, $g \neq 0$, these conditions are equivalent:

- $H_{\mu'}(g)$ is a **unit** in $\mathcal{G}r(\mu')$
- $g \sim_{\mu'} h$ for some $h \in K[x]$ with $\boxed{\deg h < \deg \phi}$
- $S_\nu(g)$ is a point on the ordinate axis



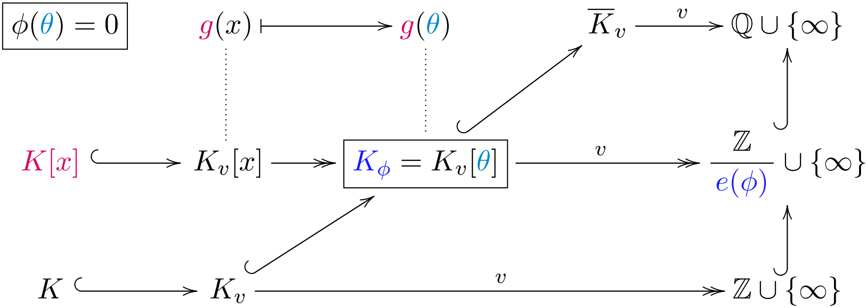
Non-equivalent conditions

$\phi \nmid_\mu g \quad \Longrightarrow \quad H_{\mu'}(g)$ is a **unit** in $\mathcal{G}r(\mu') \quad \Longrightarrow \quad \phi \nmid_{\mu'} g$

Pseudo-valuation on $K[x]$ attached to a key polynomial

Take $\mu \in \mathbb{V}$ and $\boxed{\phi \in KP(\mu)}$

The polynomial ϕ is **irreducible** over the completion K_v



Pseudo-valuation on $K[x]$ augmenting μ

$$g = g_0 + g_1\,\phi + \cdots\cdots \longmapsto \boxed{v(g(\theta))} = \mu(g_0) \geq \mu(g)$$

Some new info on key polynomials

Take $\mu \in \mathbb{V}$ and $\boxed{\phi \in KP(\mu)}$

The polynomial ϕ has **integral coefficients** : $\phi \in \mathcal{O}[x]$

The residue field $\mathbb{F}_\phi$

$$\begin{array}{ccccc}
K & \hookrightarrow & K_v & \hookrightarrow & K_\phi \\
 & & & & \\
\mathfrak{m} & \hookrightarrow & \mathfrak{m}_v & \hookrightarrow & \mathfrak{m}_\phi \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O} & \hookrightarrow & \mathcal{O}_v & \hookrightarrow & \mathcal{O}_\phi \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{F} & \xrightarrow{\simeq} & \mathbb{F}_v & \hookrightarrow & \mathbb{F}_\phi
\end{array}$$

$$\deg \phi = [K_\phi : K_v] = e(\phi)\, f(\phi)$$

$$f(\phi) := [\mathbb{F}_\phi : \mathbb{F}_v]$$

$$v(K_\phi^*) = \frac{\mathbb{Z}}{e(\phi)} \subset \Gamma(\mu) = \frac{\mathbb{Z}}{e(\mu)}$$

Some new info on key polynomials

The *residual ideal*

The **kernel** of the (onto) ring homomorphism

$$\begin{array}{rcl} \Delta(\mu) & \twoheadrightarrow & \mathbb{F}_\phi \\ g(x) + \mathcal{P}_0^+ & \longmapsto & g(\theta) + \mathfrak{m}_\phi \end{array}$$

is (the **maximal** ideal)

$$\Delta(\mu) \cap \boxed{H_\mu(\phi)\,\mathcal{G}r(\mu)}$$

Canonical embedding

$$\begin{array}{ccc} \Delta(\mu) & \twoheadrightarrow & \mathbb{F}_\phi \\ & \searrow & \big\downarrow \\ & & \Delta(\mu') \end{array}$$

for any augmentation $\boxed{\mu \xrightarrow{\phi,\,\nu} \mu'}$

Inductive valuations in $\mathbb{V}$

$$\mu_0 \xrightarrow{\phi_1\,,\,\nu_1} \mu_1 \xrightarrow{\phi_2\,,\,\nu_2} \cdots\cdots\cdots \xrightarrow{\phi_r\,,\,\nu_r} \mu_r$$

$$\boxed{\phi_i \in KP(\mu_{i-1})} \qquad \boxed{\nu_i \in \mathbb{Q}_{>0}}$$

Some attached numerical data

$$1 = \deg\phi_0 \mid \deg\phi_1 \mid \cdots\cdots\cdots \mid \deg\phi_r$$

$$\phi_0 := x$$

$$0 = C(\mu_0) < C(\mu_1) < \cdots\cdots < C(\mu_r)$$

Sharp **upper bounds** for the *normalized* values

$$C(\mu_i) := \boxed{\frac{\mu_i(\phi_i)}{\deg\phi_i}} \geq \frac{\mu_i(g)}{\deg g}$$ for all $g \in K[x] \setminus K$ **monic**

Inductive valuations in $\mathbb{V}$

$$\mu_0 \xrightarrow{\ \phi_1\,,\ \nu_1\ } \mu_1 \xrightarrow{\ \phi_2\,,\ \nu_2\ } \cdots\cdots\cdots \xrightarrow{\ \phi_r\,,\ \nu_r\ } \mu_r$$

Some attached algebraic data

$$\begin{array}{ccccccc}
 & \mathcal{G}r(\mu_0) & \longrightarrow & \mathcal{G}r(\mu_1) & \longrightarrow \cdots\cdots \longrightarrow & \mathcal{G}r(\mu_r) \\
 & \uparrow & & \uparrow & & \uparrow \\
\mathbb{F}[y] \simeq & \Delta_0 & \longrightarrow & \Delta_1 & \longrightarrow \cdots\cdots \longrightarrow & \Delta_r \\
 & \uparrow & & \uparrow & & \uparrow \\
\mathbb{F} = & \mathbb{F}_0 & \hookrightarrow & \mathbb{F}_1 & \hookrightarrow \cdots\cdots \hookrightarrow & \mathbb{F}_r
\end{array}$$

$$\begin{array}{ccc}
\Delta_{i-1} & \twoheadrightarrow & \mathbb{F}_{\phi_i} \\
 & \searrow & \downarrow \wr \\
 & & \mathbb{F}_i \subset \Delta_i
\end{array}$$

$$f_i := [\mathbb{F}_{i+1} : \mathbb{F}_i]$$

$$\boxed{f(\phi_i) = f_0 \cdots f_{i-1}}$$

Inductive valuations in $\mathbb{V}$

$$\mu_0 \xrightarrow{\phi_1\,,\,\nu_1} \mu_1 \xrightarrow{\phi_2\,,\,\nu_2} \cdots\cdots\cdots \xrightarrow{\phi_r\,,\,\nu_r} \mu_r$$

$$\boxed{\mu_0 \le \mu_1 \le \cdots\cdots\cdots \le \mu_r}$$

Sharp monotonicity

Let $g \in K[x]$ satisfy $\boxed{\mu_i(g) = \mu_{i+1}(g)}$ for some $0 \le i < r$. Then,

$$\mu_i(g) = \mu_{i+1}(g) = \cdots\cdots\cdots = \mu_r(g)$$

MacLane chains

Inductive chains in $\mathbb{V}$ (as above) satisfying, for $1 \le i < r$,

$$\boxed{\mu_i(\phi_i) = \mu_{i+1}(\phi_i)}$$

Optimal MacLane chains

Particular case: $\deg \phi_1 < \deg \phi_2 < \cdots\cdots\cdots < \deg \phi_r$

Inductive valuations in $\mathbb{V}$

Uniqueness of **optimal** MacLane chains

Every **inductive** valuation $\mu \in \mathbb{V}$ has an **optimal** MacLane chain

$$\mu_0 \xrightarrow{\phi_1\,,\,\nu_1} \mu_1 \xrightarrow{\phi_2\,,\,\nu_2} \cdots\cdots\cdots \xrightarrow{\phi_r\,,\,\nu_r} \mu_r = \mu$$

Any other **optimal** MacLane chain

$$\mu_0 \xrightarrow{\phi_1^*\,,\,\nu_1^*} \mu_1^* \xrightarrow{\phi_2^*\,,\,\nu_2^*} \cdots\cdots\cdots \xrightarrow{\phi_t^*\,,\,\nu_t^*} \mu_t^*$$

satisfies $\boxed{\mu_r = \mu_t^*}$ if and only if

$$r = t \quad \text{and} \quad \left\{ \begin{array}{c} \nu_i = \nu_i^* \\ \deg \phi_i = \deg \phi_i^* \\ \mu_i(\phi_i) = \mu_i(\phi_i^*) \end{array} \right\} \quad \text{for} \quad 1 \le i \le r$$

Moreover, in this case, $\boxed{\mu_i = \mu_i^*}$ for $1 \le i \le r$

Inductive valuations in $\mathbb{V}$

$$\mu_0 \xrightarrow{\ \phi_1\,,\ \nu_1\ } \mu_1 \xrightarrow{\ \phi_2\,,\ \nu_2\ } \cdots\cdots\cdots \xrightarrow{\ \phi_r\,,\ \nu_r\ } \mu_r$$

Value groups in a **MacLane chain**

$$\mathbb{Z} = \Gamma_0 \subset \Gamma_1 \subset \cdots\cdots\cdots \subset \Gamma_r$$

$$\boxed{\Gamma_i = \Gamma_{i-1} + \nu_i\,\mathbb{Z}} \qquad \frac{\mathbb{Z}}{e(\mu_{i-1})} = \Gamma_{i-1} = \frac{\mathbb{Z}}{e(\phi_i)}$$

Some **basic integers** attached to a MacLane chain

$$\boxed{\mathbb{Z} = e_i\,\mathbb{Z} + h_i\,\mathbb{Z}}$$

$$e_i := [\Gamma_i : \Gamma_{i-1}] = \frac{e(\mu_i)}{e(\mu_{i-1})} \qquad h_i := e(\mu_i)\,\nu_i$$

$$e_0 := 1 \qquad h_0 := \nu_0 := 0$$

$$\boxed{e(\phi_i) = e_0 \cdots e_{i-1}} \qquad \boxed{f(\phi_i) = f_0 \cdots f_{i-1}} \qquad \deg\phi_i = e_{i-1}\,f_{i-1}\deg\phi_{i-1}$$

Residual polynomial operator attached to a MacLane chain

$$\Delta_0 = \mathcal{O}[x]/\mathfrak{m}[x] \simeq \mathbb{F}[y]$$

$$H_{\mu_0}(h) \longleftrightarrow \overline{h}$$

The *special* trivial case $r = 0$

$$R_0 : K[x] \longrightarrow \mathcal{O}[x] \xrightarrow{H_{\mu_0}} \Delta_0 \overset{\simeq}{\longleftrightarrow} \mathbb{F}[y]$$

$$x \longmapsto y_0 \longleftrightarrow y$$

$$0 \neq g \longmapsto g/\pi^{\alpha} \longmapsto H_{\mu_0}(g)/p_0^{\alpha} \longleftrightarrow \boxed{\overline{g/\pi^{\alpha}}}$$

$$\alpha = \mu_0(g)$$

$$\boxed{H_{\mu_0}(g) = p_0^{\alpha}\, R_0(g)(y_0)}$$

$$\Delta_0 = \mathbb{F}[y_0]$$

$$\mathcal{G}r(\mu_0) = \mathbb{F}[y_0, p_0, p_0^{-1}]$$

Residual polynomial operator attached to a MacLane chain

$$\mu_0 \xrightarrow{\phi_1\,,\,\nu_1} \mu_1 \xrightarrow{\phi_2\,,\,\nu_2} \cdots\cdots\cdots \xrightarrow{\phi_r\,,\,\nu_r} \mu_r$$

Montes operator for $r \geq 1$

$$R_r \,:\, K[x] \longrightarrow \mathbb{F}_r[y]$$

$$\boxed{H_{\mu_r}(g) \;=\; x_r^{s_r(g)}\; p_r^{u_r(g)}\; R_r(g)(y_r)}$$

p_r is a certain *key* **unit** in $\mathcal{G}r(\mu_r)$

x_r is a certain *key* **prime** associate to $H_{\mu_r}(\phi_r)$ in $\mathcal{G}r(\mu_r)$

$y_r \;=\; x_r^{e_r}\, p_r^{-h_r}$

$$\Delta_r \;=\; \mathbb{F}_r[\,y_r]$$

$$\mathcal{G}r(\mu_r) \;=\; \mathbb{F}_r[\,y_r\,,\,p_r\,,p_r^{-1}]\,[x_r]$$

(y_r and p_r algebraically independent over $\mathbb{F}_r$)

Residual polynomial operator attached to a MacLane chain

$$\mu_0 \xrightarrow{\phi_1\,,\,\nu_1} \mu_1 \xrightarrow{\phi_2\,,\,\nu_2} \cdots\cdots\cdots \xrightarrow{\phi_r\,,\,\nu_r} \mu_r$$

Montes operator for $r \geq 1$

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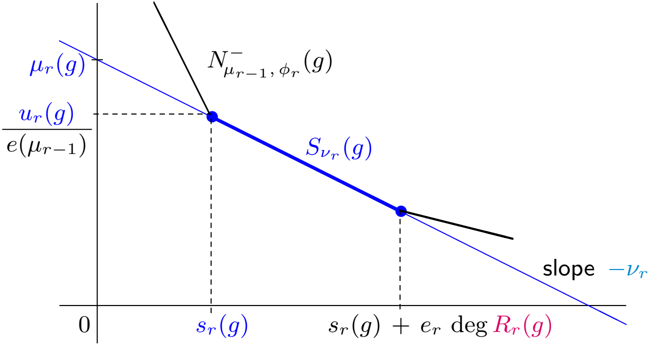


Residual polynomial operator attached to a MacLane chain

Some nice *desired* properties of R_r for $r \geq 1$

- For $g, h \in K[x]$, $g, h \neq 0$,

$$\boxed{g \sim_{\mu_r} h} \iff S_{\nu_r}(g) = S_{\nu_r}(h) \quad \text{and} \quad R_r(g) = R_r(h)$$

- For $g, h \in K[x]$, $\boxed{R_r(gh) = R_r(g)\, R_r(h)}$

- For $g \in K[x]$ with $\deg R_r(g) > 0$,

$$\boxed{g \quad \mu_r\text{-irreducible}} \iff s_r(g) = 0 \quad \text{and} \quad R_r(g) \text{ irreducible in } \mathbb{F}_r[y]$$

- The polynomial $R_{r-1}(\phi_r)$ is irreducible in $\mathbb{F}_{r-1}[y]$ and it defines the extension $\mathbb{F}_r / \mathbb{F}_{r-1}$

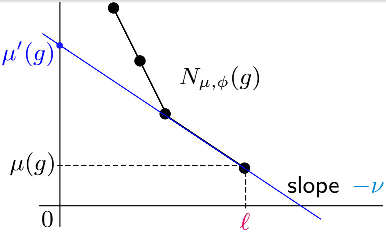
Bonus track...?

Augmentations of a valuation μ on $K(x)$

$$g = g_0 + g_1\,\phi + \cdots + g_\ell\,\phi^\ell \in K[x] \qquad (\deg g_s < \deg \phi, \quad g_\ell \neq 0)$$

A practical criterion for μ'-minimality

g is μ'-**minimal** $\iff$ $\deg g_\ell = 0$ and $\mu'(g) = \mu'\left(g_\ell\,\phi^\ell\right)$



Corollary

Let $g \in K[x] \setminus K$ be μ'-**minimal**. Then, $\deg \phi$ divides $\deg g$.

Moreover, $C(\mu') := \dfrac{\mu'(\phi)}{\deg \phi} = \dfrac{\mu'(g)}{\deg g}$ if g is **monic**.