Una Passejada Formal
A Formal Promenade.

To Enric Nart, in his 60th birthday's

Xavier Xarles

29 de Gener de 2015

Formal Groups

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$d =$ the dimension of F.

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- The Lubin-Tate formal groups.

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Note that the existence of inverses can be deduced from the axioms.

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Note that the isomorphism $\widehat{A} \cong R[[X_1, \ldots, X_d]]$ is determined giving a basis of $\widehat{I}/\widehat{I}^2 \cong \omega_{T/R}$, the cotangent module.

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- The formal group associated to an elliptic curve E is the formal completion of E.

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Hence all formal group over R are (strongly) isomorphic to $\widehat{\mathbb{G}a}$.

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Theorem (Honda,1970)

Consider the sequence $I = A_1, A_2, \ldots, A_n, \cdots \in M_d(R)$ such that

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be the Dirichlet L-series associated d.

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Moreover F_L is strongly isomorphic over $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ to $X + Y + \sqrt{d}XY$.

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Theorem (Demchenko,Gurevich,Xarles (2010))

If $K/\mathbb{Q}$ is tamely ramified, the result is true for $S = \emptyset$ (by using the "natural" A_p for the primes of bad reduction).

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Then the local L-factor of A relative to F has the form

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To Enric Nart, in his 60th birthday's

Xavier Xarles

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Then there is an isomorphism between $\widehat{\mathcal{A}}$ and $\widehat{L}$ over $\mathbb{Z}_S$.

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Theorem (Demchenko, Gurevich, Xarles(2010))

Let T and T' be tori split under tamely ramified abelian extension of $\mathbb{Q}$. Then the natural map

$$\mathrm{Hom}(T, T') \cong \mathrm{Hom}(\widehat{\mathcal{T}}, \widehat{\mathcal{T}'})$$

is an isomorphism.

Moltes Felicitats!

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Angela

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Núria

Moltes Felicitats!

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