Computation of Framed Deformation Functors

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Notation

- p rational prime;
- k finite field of characteristic p;
- Ar category of local artinian complete $W(k)$-algebras A with surjective homomorphism $\pi_A : A \to k$ and maximal ideal m_A;
- $\hat{Ar}$ category of local noetherian complete $W(k)$-algebras A with surjective homomorphism $\pi_A : A \to k$ and maximal ideal m_A;
- S finite set of primes of $\mathbb{Q}$ including p and the infinite archimedean prime;
- $G = G_{\mathbb{Q},S}$ the Galois group of the maximal extension unramified outside S;
- $\bar{\rho} : G \to GL_2(k)$ Galois representation;
- $V_{\bar{\rho}}$ the G-module associated to $\bar{\rho}$;
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Deformations

Definition

Let $A \in Ar$. A deformation of $V_{\bar{\rho}}$ to A is a pair (V_A, ι_A), where

- V_A is a free $A[G]$-module provided with a G-continous action;
- $\iota_A : V_A \otimes k \simeq V_{\bar{\rho}}$.

Definition

Let β be a k-basis of $V_{\bar{\rho}}$. A framed deformation of the pair $(V_{\bar{\rho}}, \beta)$ to A is a triple (V_A, ι_A, β_A), where

- (V_A, ι_A) is a deformation of $V_{\bar{\rho}}$ to A;
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Deformation functors

Definition

The deformation functor $F_{\bar{\rho}} : Ar \to Sets$ attached to $\bar{\rho}$ is defined as

$$F_{\bar{\rho}}(A) = \{\text{isomorphism classes of deformations of } V_{\bar{\rho}} \text{ to } A\} \tag{1}$$

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The framed deformation functor $F_{\bar{\rho}}^{\square} : Ar \to Sets$ attached to $\bar{\rho}$ is defined as

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Representability

Theorem (Mazur)

- *$F_{\bar{\rho}}^{\square}$ is pro-representable, that is, there exists $R^{\square} = R_{\bar{\rho}}^{\square} \in \hat{A}r$ such that*
$$F_{\bar{\rho}}(A) = Hom_{W(k)}(R^{\square}, A); \tag{3}$$
- *If $End_G(V_{\bar{\rho}}) = k$, then $F_{\bar{\rho}}$ is pro-representable by a ring $R = R_{\bar{\rho}} \in \hat{A}r$.*

The representing algebras R and $R^{\square}$ are called the universal deformation ring and the universal framed deformation ring attached to $\bar{\rho}$ respectively.

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The representing algebras R and $R^{\square}$ are called the universal deformation ring and the universal framed deformation ring attached to $\bar{\rho}$ respectively.

The tangent space

Let ϵ be an element such that $\epsilon^2 = 0$. Then we can define the following.

Definition

Let $F_{\bar{\rho}}$ be a deformation functor. The tangent space of $F_{\bar{\rho}}$ is the set

$$F_{\bar{\rho}}(k[\epsilon]). \tag{4}$$

It has a natural structure of k-vector space.

Lemma

- $F_{\bar{\rho}}(k[\epsilon]) \simeq H^1(G, Ad(\bar{\rho})) \simeq Ext^1_{k[G]}(V_{\bar{\rho}}, V_{\bar{\rho}})$;
- $dimF^{\square}_{\bar{\rho}}(k[\epsilon]) = dimF_{\bar{\rho}}(k[\epsilon]) + dimAd(\bar{\rho}) - dimH^0(G, Ad(\bar{\rho}))$.

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Local-to-global arguments

For every $v \in S$ we assume the following:

- we fix an algebraic closure $\bar{\mathbb{Q}}_v$ of $\mathbb{Q}_v$;
- we fix an inclusion morphism $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_v$;
- Let G_v be the absolute Galois group of $\mathbb{Q}_v$. We can see it as a subgroup of G via the inclusion morphism;
- we put $\bar{\rho}_v = \bar{\rho}|_{G_v}$.

Then we can consider the local deformation functor $F_{\bar{\rho}_v}$ attached to $\bar{\rho}_v$, with some additional condition, and compute its (framed and unframed) universal deformation ring. Then we want to recover the universal deformation ring of $\bar{\rho}$ from this local data.

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An explicit problem

Let E be an elliptic curve over $\mathbb{Q}$ with good supersingular reduction in p and good reduction in all but one prime $\ell \neq p$, where it has semistable reduction. We take $S = \{\ell, p, \infty\}$. Let

$$\bar{\rho} : G \to GL_2(\bar{F}_p) \quad (5)$$

be the Galois representation attached to the p-torsion points of E. Such a representation has determinant equal to the cyclotomic character χ.

Moreover we ask that $Ext^1_{D,p}(E[p], E[p]) = 0$, where D is the subcategory of p-power order group schemes having semistable G-action in ℓ and the subscript p means that we consider only extensions which are killed-by-p.

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We impose the following local conditions:

- $\bar{\rho}$ is odd, that is $det(\bar{\rho}(\gamma)) = -1$ for γ a complex conjugation;
- $(\bar{\rho}(\sigma) - id)^2 = 0$ for every $\sigma \in I_\ell$;
- $\bar{\rho}_p$ is flat, that is, $V_{\bar{\rho}_p}$ is the generic fiber of a finite flat group scheme over $\mathbb{Z}_p$.

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Case $v = \infty$

Let $\rho_\infty : G_\infty \to GL_2(k)$ be the restriction of $\bar{\rho}$ at the archimedean prime. Since $G_\infty \simeq \mathbb{Z}/2\mathbb{Z}$ a framed deformation is uniquely determined by a lift of $\bar{\rho}_\infty(\gamma)$. Such a lift has the form:

$$\rho_\infty(\gamma) = \bar{\rho}_\infty(\gamma) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6}$$

In order to obtain an odd lift, we need to impose $det(\bar{\rho}_\infty(\gamma)) = -1$ and $Tr(\bar{\rho}_\infty(\gamma)) = 0$
In the end we have $R_\infty^{\square,\chi} \simeq W(k)[[x_1, x_2]]$.

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Case $v = p$

Definition

Let ϕ be the absolute Frobenius morphism. A Fontaine-Laffaille module is $\mathbb{Z}_p$-module M of finite length provided with a filtration $\{M_i\}$ of $\mathbb{Z}_p$-submodule such that:

1. $M_0 = M$;
2. $M_j = 0$ for a finite index j;
3. for every index i there exist a ϕ-semilinear map $\phi_i : M_i \to M$ such that $\phi_i(x) = p\phi_{i+1}(x)$.

A Fontaine-Laffaille module is called connected if ϕ_0 is nilpotent.

Theorem (Fontaine-Laffaille)

There exists a faithful exact contravariant functor

$$MF^{j}_{tor} \to Rep^{fl}_{\mathbb{Z}_p}(G_p), \tag{7}$$

between torsion Fontaine-Laffaille modules of length j and flat representations of G with values in $\mathbb{Z}_p$. This functor is fully faithful if $j < p$ and becomes fully faithful when restricted to the subactegory of connected modules if $j = p$.

Theorem (Ramakrishna)

Let $\bar{\rho} : G_p \to GL_2(k)$ be a flat representation with $End_G(\bar{\rho}) = k$ and $det(\bar{\rho}) = \chi$. Then

$$R_{\bar{\rho}}^{fl} \simeq W(k)[[x_1, x_2]]. \tag{8}$$

Theorem (Conrad)

Let $\bar{\rho}$ be as before. Then

$$R_{\bar{\rho}}^{fl,\chi} \simeq W(k)[[x]]. \tag{9}$$

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Case $v = \ell$

Definition

A representation $\bar{\rho}_\ell : G_\ell \to GL_2(k)$ is called of Steinberg type if it is a non-trivial extension of an unramified character λ of G_ℓ by $\lambda(1) = \lambda \otimes \chi_p$.

Let $L_{\bar{\rho}_\ell} : Ar \to Sets$ be a subfunctor of $R^{\chi}_{\bar{\rho}_\ell}$ defined by $L_{\bar{\rho}_\ell}(A) = (V_A, \iota_A, L_A)$, where:

- (V_A, ι_A) is a deformation of $\bar{\rho}_\ell$;
- L_A is a 1-dimensional submodule of V_A with G_ℓ-action provided by χ_p.

$L_{\bar{\rho}_\ell}$ is the subfunctor corresponding to lift of Steinberg type.

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Theorem (Kisin)

$L^{\square}_{\bar{\rho}_\ell}$ is representable by a ring $R^{\square,1,\chi}_{\bar{\rho}_\ell}$ of Krull dimension 4.

Geometric deformation functors

Definition

A deformation functor $F_{\bar{\rho}}$ is called geometric if, for every $v \in S$, the local framed deformation ring $R_v^{\square,\chi}$ satisfies:

$$dim R_v^{\square,\chi}[1/p] = \begin{cases} 3 & \text{if } v \neq p, \infty \\ 4 & \text{if } v = p \\ 2 & \text{if } v = \infty \end{cases} \qquad (10)$$

Theorem

If $F_{\bar{\rho}}$ is a geometric deformation functor, then $dim R_{\bar{\rho}} \geq 1$.

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The Main Theorem

Let

$$\bar{\rho} = \bar{\rho}_1 \oplus \ldots \oplus \bar{\rho}_n : G \to GL_{2n}(k), \tag{11}$$

where each $\bar{\rho}_i$ is a 2-dimensional Galois representation with values in k and such that $V_{\bar{\rho}_i}$ is the generic fiber of a finite flat group scheme contained in a subcategory D of p-group schemes. Up to reordering indexes, we may assume the first r representations being non-isomorphic and the others $n-r$ isomorphic to one of the previous. Then we rewrite

$$\bar{\rho} = \oplus_{i=1}^{r} \bar{\rho}_i^{e_i}. \tag{12}$$

Let $F_{\bar{\rho}_i,D}$ be the deformation functor associated to $\bar{\rho}_i$ with the local conditions defined previously and $F_{\bar{\rho},D}$ the deformation functor attached to $\bar{\rho}$ with the local conditions applying to each i.

Theorem (Main Theorem)

Suppose:

1. $Ext^1_{D,p}(V_{\bar{\rho}_i}, V_{\bar{\rho}_j}) = 0$ *for every* $i, j = 1, \ldots, r$;
2. $Hom_G(V_{\bar{\rho}_i}, V_{\bar{\rho}_j}) = \begin{cases} k & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

Then the functor $F^{\square}_{\bar{\rho},\underline{D}}$ *is represented by a power series ring over* $W(k)$ *in* N *variables, where*

$$N = 4n^2 - \sum_{i=1}^{r} e_i^2. \tag{13}$$

1^{st} step

Lemma

$F_{\bar{\rho}_i}$ is a geometric deformation funcor.

It follows immediatly from the previous results.

2^{nd} step

Lemma (Kisin)

$F_{\bar{\rho}_i}$ *geometric* $+$ $Ext_{D,p}(V_{\bar{\rho}_i}, V_{\bar{\rho}_i}) = 0 \Rightarrow \bar{\rho}_i$ *admits a p-adic lift* ρ_i *to* $W(k)$.

Let

$$\bar{T} = \begin{pmatrix} \begin{pmatrix} \bar{\rho}_1 & & \\ & \ddots & \\ & & \bar{\rho}_1 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \bar{\rho}_r & & \\ & \ddots & \\ & & \bar{\rho}_r \end{pmatrix} \end{pmatrix}. \qquad (14)$$

be the matrix associated to $\bar{\rho}$ and T the matrix obtained replacing each $\bar{\rho}_i$ with ρ_i.

3^{rd} step

Deform $T \Rightarrow \hat{T} = T(I_{2n} + M)$, where M is a $2n \times 2n$ matrix having a variable $x_{i,j}$ as i, j-th element.

$\hat{T}$ has entries in $\hat{R} = W(k)[[x_{i,j}]]$.

4^{th} step

Deform $\hat{T} \Rightarrow \tilde{T} = \hat{T}(I_{2n} + Y)$, where Y is a matrix commuting with $\tilde{T}$ and killing some of the $x_{i,j}$.

In order to respect the mutual endomosphism condition, we take $Y = diag[Y_1, \ldots, Y_n]$, where $Y_i = A_i \otimes I_2$. We need to choose properly the entries of the matrices A_i. We put

$$a_{ist} = \frac{-x_{h_i+2s,h_i+2t}}{1 + x_{h_i+2s,h_i+2t}}, \tag{15}$$

where $h_i = \sum_{j=1}^{i} 2e_j$.

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The resulting deformation has the form $\tilde{T} = T(I_{2n} + \tilde{M})$, where $\tilde{M}$ is a matrix of variables $\tilde{x}_{u,v}$ given by

$$\tilde{x}_{u,v} = \begin{cases} 0 & \text{if } u = h_i + 2s,\ \ v = h_i + 2t \\ \frac{x_{u,v}}{1+x_{h_i+2s,h_i+2t}} & \text{otherwise} \end{cases} \tag{16}$$

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5^{th} step

Compute the dimension of the tangent space of $F_{\bar{\rho}}^{\square}$ using the exact sequence

$$0 \to F_{\bar{\rho},D}(k[\epsilon]) \to F_{\bar{\rho},D}^{\square}(k[\epsilon]) \to Ad(\bar{\rho})/Ad(\bar{\rho})^{G} \to 0. \qquad (18)$$

This dimension is exactly $4n^2 - \sum_{i=1}^{r} e_i^2 = N$.

6^{th} step: Universality

Let $R^{\square}_{\bar{\rho},D}$ be the universal deformation ring. Consider the diagram

$$\begin{array}{ccc} W(k)[[x_1,\dots,x_N]] & \xrightarrow{\pi_1} & R^{\square}_{\bar{\rho},D} \\ & \searrow^{\pi_2} & \downarrow \pi \\ & & \tilde{R} \simeq W(k)[[y_1,\dots,y_N]]. \end{array}$$

where π is the unique map associated to the deformation $\tilde{T}$ by universality. It is enough to prove that π is sujective (and therefore an isomorphism).

Apply the functor $Hom(., k[\epsilon])$ and pass to the mod p tangent spaces. There the map induced by π is injective, because of the trivial centralizer hypothesis.

This concludes the proof of the Main Theorem.

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