p-ADIC ANALOGUES OF THE BSD CONJECTURE
(À LA MTT)

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Summary

Quick overview

1. Introduction
2. Notations
3. L-functions and distributions
4. The extended Mordell-Weil group

Schedule of the seminar

p-adic methods for elliptic curves

- Two related topics in the STNB: Hida families and p-adic methods for elliptic curves.
- In many recent results, Hida families turn out to be essential tools for proving Gross-Zagier type formulas, results about special values of p-adic L-series...
- In this part we focus on p-adic methods and relation with BSD conjecuture:
  1. p-adic analogues of BSD conjectures (R., Tuesday).
  2. The exceptional zero conjecture (Barrera, Wednesday).
  3. A second derivative result for p-adic L-functions (de Vera, Wednesay).
  4. The MTT conjecture in the rank one setting (Rotger, Thursday).
  5. The rationality of traces of Stark-Heegner points (Masdeu, Friday).

Historical background

Back to 1986

- 1986: Mazur, Tate, Teitelbaum publish “On p-adic analogues of the conjectures of Birch and Swinertonn-Dyer”: since the p-adic analogue of the Hasse-Weill L-function had been defined, and also p-adic theories analogous to the theory of canonical height had been introduced, “it seemed to us to be an appropriate time to embark on the project of formulating a p-adic analogue of BSD [...] The project has proved to be anything but routine.
- Appearance of a p-adic multiplier, local term not equal to any recognizable Euler fact. It can vanish at the central point (exceptional case). We expect in this case the order of vanishing to be one greater than the classical one.
- This agrees with the fact that, when E split multiplicative at p, we have extended Mordell-Weil group, whose rank is one greater than the usual Mordell-Weil group.

Aims for the talk

And more history

- We will introduce both the L-function and the extended Mordell-Weil group.
- In the split multiplicative case: we will formulate p-adic BSD and MTT conjecture: it expresses $L'_p(E,1)$ as a product of $\mathcal{L} := \log(q)/\operatorname{ord}_p(q)$ and the algebraic part of $L(E,1)$.
- In Daniel's talk, we will see how Greenberg and Stevens proved a special case using Hida's theory of p-adic families of ordinary eigenforms.
- In Victor's talk, we will see Vererucci's results when the rank of the Mordell-Weil group of $E/\mathbb{Q}$ is 1. Interplay between Heegner points and Beilinson-Kato elements. It uses p-adic variation of modular forms.

Preliminaries

Actions

- For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$, $\rho(A) = \frac{(\det A)^{1/2}}{cz+d}$.
- Action in $S_k(= \oplus S_k(N, \epsilon))$: $(f|A)(\tau) = \rho(A)^k f(A\tau)$.
- $P_k(\mathbb{C})$ polynomials of degree $\leq k-2$. Action $(P|A)(z) = \rho(A)^{2-k} P(A(z))$.
- $\Delta = \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$. Since $d(A\tau) = d\tau \rho(A)^2$,

$$(f|A)(P|A)d\tau = f(A(\tau))P(A(\tau))d(A\tau).$$

- Application $\Phi : S_k \to \mathrm{Hom}_{\mathbb{Z}}(\Delta), P_k(\mathbb{C})^*)$ or alternatively $\phi : S_k \times P_k(\mathbb{C}) \times \mathbb{P}^1(\mathbb{Q}) \to \mathbb{C}$, where $\phi(f, P, r)$ is

$$2\pi i \int_{\infty}^{r} f(z)P(z)dz = \begin{cases} 2\pi \int_0^{\infty} f(r+it)P(r+it)dt, & \text{if } r \in \mathbb{Q}, \\ 0, & \text{if } r = \infty. \end{cases}$$

More preliminaries

Modular symbols

- A modular integral is a mapping $\Phi : S_k \times P_k(\mathbb{C}) \times \mathbb{P}^1(\mathbb{Q}) \to V$, ($V$ complex vector space) $\mathbb{C}$-bilinear in f and P such that

$$\Phi(f|A, P|A, r) = \Phi(f, P, A(r)) - \Phi(f, P, A(\infty)).$$

- It can be used to define a modular symbol λ. For $a, m \in \mathbb{Q}$, $m > 0$, $f \in S_k$ and $P \in P_k(\mathbb{C})$, let

$$\lambda(f, P; a, m) := \Phi(f, P(mz + a), -a/m)$$

$$= m^{k/2-1}\Phi\left(f, P\middle| \begin{pmatrix} m & a \\ 0 & 1 \end{pmatrix}, -a/m\right)$$

$$= m^{k/2-1}\Phi\left(f\middle| \begin{pmatrix} 1 & -a \\ 0 & m \end{pmatrix}\right), P, 0).$$

First results

Classical stuff

- $L_f \subset V$ $\mathbb{Z}$-module generated by the image of ϕ_f.
- The $\mathbb{Z}$-module L_f is the $\mathbb{Z}[\epsilon]$-submodule of V generated by $\Phi(f, z^i, A_j(\infty)) - \Phi(f, z^i, A_j(0))$ for $0 \leq i \leq k-2$ and with $\mathrm{SL}_2(\mathbb{Z}) = \sqcup \Gamma_0(N) \cdot A_j$.
- $\lambda(f, P; a, m)$ is $\mathbb{C}$-bilinear in (f, P). For fixed f, $\lambda_f(P; a, m)$ takes values in L_f. For fixed f, P, $\lambda_{f,P}(a, m)$ depends only on a modulo m.
- Next aim: relate modular symbols with the values of $L(f, s)$.

Modular symbols and values of $L(f,s)$

- For $f=\sum_{n\geq 1} a_n q^n \in S_k(\epsilon)$,

$$L(f,s) := \sum_{n\geq 1} a_n n^{-s} = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f(it)t^s(dt/t).$$

- $\lambda(f,z^n,0,1)=\phi(f,z^n,0)=i^n\frac{n!}{(2\pi)^n}L(f,n+1)$ for $0\leq n\leq k-2$.
- When χ is primitive we have Birch's lemma

$$f_{\bar{\chi}}(z)=\frac{1}{\tau(\chi)}\sum_{a \bmod m} \chi(a)f(z+a/m).$$

- Further, relation between $\phi(f_{\bar{\chi}},P,r)$ and $\phi(f,P(z-a/m),r+a/m)$.
- Remarkable result

$$L(f_{\bar{\chi}},n+1)=\frac{1}{n!}\frac{(-2\pi i)^n}{m^{n+1}}\tau(\bar{\chi})\sum_{a \bmod m} \chi(a)\lambda(f,z^n;a,m).$$

Action of operators

Hecke and family

- Recall: $f \in S_k(N, \epsilon)$,
$f|T_l = l^{k/2-1}\Big(\sum_{u=0}^{l-1} f\Big|\begin{pmatrix} 1 & u \\ 0 & l \end{pmatrix} + \epsilon(l)f\Big|\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}\Big)$.
- $\lambda(f|T_l, P; a, m) = \sum_{u=0}^{l-1} \lambda(f, P; a-um, lm) + \epsilon(l)l^{k-2}\lambda(f, P; a, m/l)$.
- $N = Q \cdot Q', \epsilon = \epsilon_Q \epsilon_{Q'}$. $w_Q(f) \in S_k(N, \epsilon_Q^{-1}\epsilon_{Q'})$.
- $w_Q^2(f) = (-1)^k \epsilon_{Q'}^{-1}(-Q)f$.
- Functional equation for modular symbols: a, m relatively prime, $m > 0$, $(m, Q) = 1$, $Q'|m$. a' such that $a'aQ \equiv -1$ modulo m.

$$\lambda(f, P; a, m) = -\epsilon_Q(-m)\epsilon_{Q'}^{-1}(-a)\lambda\Big(w_Q(f), P\Big|\begin{pmatrix} 0 & -1 \\ Q & 0 \end{pmatrix}; a', m\Big).$$

Distributions and integrals

Towards Vishik, Amice,Vélu

- p prime number, $f \in S_k(N, \epsilon)$ eigenform for T_p with eigenvalue a_p. Assume $X^2 - a_p X + \epsilon(p)p^{k-1}$ has a non-zero root and let $v(m) = \mathrm{ord}_p(m)$.

$$\mu_{f,\alpha}(P; a, m) := \frac{1}{\alpha^{v(m)}}\lambda_{f,P}(a, m) - \frac{\epsilon(p)p^{k-2}}{\alpha^{v(m)+1}}\lambda_{f,P}(a, m/p),$$

  for $a, m \in \mathbb{Z}$ and $m > 0$.
- It satisfies the distribution property.
- $M > 0$ prime to p. $\mathbb{Z}_{p,M} = \mathbb{Z}_p \times (\mathbb{Z}/M\mathbb{Z})$. $\mathbb{Z}_{p,M}^{\times}$ is a p-adic analytic Lie group with fundamental system of open disks $D(a, \nu) := a + p^{\nu} M\mathbb{Z}_{p,M}$ for $n \geq 1$.
- $V_f := \mathbb{C}_p \otimes_{\mathbb{Q}} L_f\bar{\mathbb{Q}}$. Aim: define a V_f-valued integral $\int_U F$ where U is a compact open set of $\mathbb{Z}_{p,M}^{\times}$ and F is locally analytic.

Vishik-Amice-Vélu

Existence of the integral

We expect that $\int_{D(a,\nu)} P(x_p) = \mu_{f,\alpha}(P, a, p^\nu M)$, for $\nu, a \in \mathbb{Z}$, $\nu \geq 1$.

Theorem

Let $1 \leq h \leq k-1$, such that $X^2 - a_p X + \epsilon(p)p^{k-1}$ has a root α such that $\mathrm{ord}_p(\alpha) < h$. There exists a unique V_f-valued integral such that for $\nu \geq 1$ and $a \in \mathbb{Z}$ satisfies:

1. *It is $\mathbb{C}_p$ linear in F and finitely additive in U.*
2. $\int_{D(a,\nu)} x_p^j = \mu_{f,\alpha}(z^j; a, p^\nu M)$, *for $0 \leq j < h$.*
3. *(Divisibility property): For $n \geq 0$, $\int_{D(a,\nu)} (x-a)_p^n \in \left(\frac{p^n}{\alpha}\right)^\nu \alpha^{-1} \Omega_f$.*
4. *If $F(x) = \sum_{n\geq 0} c_n (x-a)_p^n$ converges on $D(a,\nu)$, then*

$$\int_{D(a,\nu)} F = \sum_{n\geq 0} c_n \int_{D(a,\nu)} (x-a)_p^n.$$

The p-adic L-function

Construction

- $\chi : \mathbb{Z}_{p,M}^{\times} \to \mathbb{C}_p^{\times}$ p-adic character.
- $x = \omega(x)\langle x \rangle$.
- $\chi(x) = x_p^j \psi(x)$, $0 \leq j \leq k-2$. ψ of finite order.
- $\chi_s(x) := \langle x \rangle^s = \exp(s \log\langle x \rangle) = \sum_{r=0}^{\infty} \frac{s^r}{r!} (\log\langle x \rangle)^r$.
- If α allowable p-root of f, for a p-adic character χ,

$$L_p(f, \alpha, \chi) = \int_{\mathbb{Z}_{p,M}^{\times}} \chi \, d\mu_{f,\alpha}.$$

- We define $L_p(f, \alpha, \chi, s) := L_p(f, \alpha, \chi\chi_s)$.
- $L_p(f, \alpha, \psi, s) = \sum_{r=0}^{\infty} \frac{s^r}{r!} \sum_{a \bmod p^{\nu} M} \psi(a) \int_{D(a,\nu)} (\log_p(x_p))^r$.

Towards the exceptional zero

Factorization

Let $\chi(x) = x_p^j \cdot \psi(x)$, where ψ is of conductor $m = p^\nu M$.

Proposition

$$L_p(f, \alpha, \chi) = e_p(\alpha, \chi) \cdot \frac{m^{j+1}}{(-2\pi i)^j} \cdot \frac{j!}{\tau(\bar{\psi})} \cdot L(f_{\bar{\psi}}, j+1),$$

$$e_p(\alpha, \chi) = e_p(\alpha, j, \psi) := \frac{1}{\alpha^\nu}\left(1 - \frac{\bar{\psi}(p)\epsilon(p)p^{k-2-j}}{\alpha}\right)\left(1 - \frac{\psi(p)p^j}{\alpha}\right).$$

The proof is a computation, where we introduce
$D(a, 0) = \mathbb{Z}_{p,M}^\times \cap (a + M\mathbb{Z}_{p,M})$.
$D(a, 0) = \sqcup D(b, 1)$, where $b \equiv a \bmod M$, b is not 0 modulo p and it runs over a set of such numbers modulo pM.

The p-adic L-function

Exceptional zero

- The value of the p-adic L-function at χ $L_p(f,\alpha,\chi) = L_p(f,\alpha,\omega^j\psi,j)$ is non-zero if and only if both $L(f_{\bar\psi}, j+1)$ and $e_p(\alpha,\chi)$ are non-zero.
- The pair (α, j) is exceptional if there is finite character ψ with $e_p(\alpha, x_p^j, \psi) = 0$.

Proposition

(α, j) *is exceptional if and only if:*

1. *k even, $p||N$, $\bar\epsilon(p) \neq 0$, $j = (k-2)/2$.*
2. *k odd, $p \nmid N$, $\alpha = \zeta p^{(k-1)/2}$, $j = (k-1)/2$ or $j = (k-3)/2$.*
3. *k odd, $\mathrm{ord}_p(N) = \mathrm{ord}_p(\mathrm{cond}\,\tilde\epsilon) > 0$, $a_p = \zeta_{p^{(k-1)/2}}$ and $j = (k-1)/2$.*

Order of vanishing: one greater when $e_p(\alpha, j, \psi) = 0$.

Conjectures about the order of vanishing

Exceptional zero

- $\rho_\infty(f,\psi,j)$ order of zero at $s=j+1$ of $L(f_{\bar\psi},s)$.
- $\rho_p(f,\alpha,\psi,j)$ order of zero at $s=j$ of $L_p(f,\alpha,\psi\omega^j,s)$ (or at $s=0$ of $L_p(f,\alpha,\chi,s)$, for $\chi=x_p^j\cdot\psi$).
- CONJECTURE A: $\rho_p(f,\alpha,\psi,j)=\rho_\infty(f,\psi,j)$ when $e_p(\alpha,j,\psi)\neq 0$.
- CONJECTURE B: $\rho_p(f,\alpha,\psi,j)=\rho_\infty(f,\psi,j)+1$ when $e_p(\alpha,j,\psi)=0$.

The functional equation

Come back

- $N = QQ', \epsilon = \epsilon_Q \cdot \epsilon_{Q'}$. $Q' | p^\nu M$. Operator $w_Q : S_k(N, \epsilon) \to S_k(N, \epsilon^*)$.
- f eigenvector for T_l with eigenvalue a_l, then $f^* := w_Q(f)$ eigenvalue $a_l \epsilon_Q^{-1}(l)$.
- $D(a, \nu)^*$ image under $x \mapsto -1/Qx$. $D(a, \nu)^* = D(a', \nu)$ where $aa'Q \equiv -1$ modulo $p^\nu M$.
- $\mu(f, \alpha, P; a, p^\nu M) = -\epsilon_Q(-M)\epsilon_{Q'}^{-1}(-a)\mu(f^*, \alpha^*, P^*, a', p^\nu M)$.
- $L_p(f, \alpha, \chi, s) = C \cdot L_p(f^*, \alpha^*, \epsilon_{Q'}, x_p^{k-2}\chi^{-1}, s)$.
- The sign changes from the classical to the p-adic L-function iff $e_p(\alpha, j, \psi) = 0$.

The $\mathcal{L}$-invariant of an elliptic curve over a local field

Beginning of the second part

Recall that

$$j = q^{-1} + 744 + 196884q + 21493760q^2 + \dots,$$

$$q = j^{-1} + 744j^{-2} + 750420j^{-3} + \dots.$$

- $[K : \mathbb{Q}_p] < \infty$; $E_{/K}$ non-integral j-invariant. Putting $j(E)^{-1}$ in the second series, convergent series in K with limit $q(E) \in K^\times$. It is called the multiplicative period of E.
- $\lambda : K^\times \to \mathbb{Q}_p$ continuous homomorphism. $\mathcal{L}_\lambda(E) := \lambda(q(E))/\operatorname{ord}(q(E))$.
- Typically, λ is the composition of the norm with the Iwasawa logarithm.

Motivation

Néron-Tate height

Let E an elliptic curve over $\mathbb{Q}$. Define

$$H(P) = \begin{cases} H((x(P) : z(P))) & \text{if } z(P) \neq 0 \\ 1 & \text{if } P(0:1:0). \end{cases}.$$

- $h(P) = \log H(P)$.
- $\hat{h}(P) = \lim_{n\to\infty} \frac{h(2^n P)}{4^n}$. Quadratic form.
- $\hat{h} : E(\mathbb{Q}) \to \mathbb{R}$ is called the canonical or Néron-Tate height function.
- If $\{P_1, \ldots, P_N\}$ generators of the Mordell-Weil group, the determinant of the matrix of the pairing arises in BSD.

Sigma functions

An esoteric construction

- $[K : \mathbb{Q}_p] < \infty$. $E_{/O}$ Néron model over the ring of integers. $E_{/K}$ is ordinary if the formal compoletion $E^f_{/O}$ is a formal group of height 1 or alternatively, the Néron model has multiplicative reduction or good reduction, the special fiber of which has a point of order p over the algebraic closure of the residue field.
- Choose ω regular differential. Choose t uniformizing parameter on $E^f_{/O}$.
- AIM: define a pairing analogous to the one of the Néron-Tate height: either analytically (with p-adic σ-function) or algebraically (class field theory).

More on sigma functions

Results

Define a sigma function (or the square of a sigma function when characteristic not 2). If $f \in O[[t]]$ of the form $f \equiv t^m$ modulo t^{m+1}, $D_\omega(f) = \frac{d}{\omega}\left(f^{-1}\frac{df}{\omega}\right) = -\frac{m}{t^2} + \ldots \in t^{-2}O[[t]]$.

Proposition

There is a unique formal function $\sigma_\omega^2 = \sigma_{E,\omega}^2$ on $E^f_{/O}$ whose power series expression is in $t^2 \cdot (1 + tO[[t]])$ and which satisfies:

1. *$\sigma_\omega^2(P)$ is an even function of $P \in E^f(O)$.*
2. *$D_\omega\sigma_\omega^2(P) - D_\omega\sigma_\omega^2(Q) = 2x(Q) - 2x(P)$, for $P, Q \in E^f(O)$.*

Some explicit computations can be made.
It is used for defining the canonical height.

Tate uniformization

For multiplicative reduction

- $E_{/K}$ ordinary elliptic curve with non-integral j-invariant.
- $[K' : K] \leq 2$ such that there is a rigid analytic parametrization of $E_{/K'}$ as quotient of $\mathbb{G}_{m/K'}$ by a discrete subgroup of rank one.
- For L/K' finite, $0 \to \mathbb{Z} \to L^\times \xrightarrow{i} E(L) \to 0$.
- Isomorphism between $O(L)^\times$ and $E^0(L)$.

Pairings

Reminiscences of heights

S finite set of non-archimedian primes. $E_S(K) \subset E(K)$ is the subgroup of finite index defined by: $P \in E_S(K)$ if and only if P specializes to the connected component of zero on the special fiber of the Néron model of E forr all non-archimedian v and P specializes to 0 for $v \in S$.
For $P \in E_S(K)$, define an adèle $i(P) \in K^\times \backslash \mathbb{A}_K^\times / \prod_{v \notin S} U_v$.

Proposition

There exists a bilinear symmetric pairing

$$E_S(K) \times E_S(K) \to K^\times \backslash \mathbb{A}_K^\times / \prod_{v \notin S} U_v$$

called the analytic height pairing, such that $\langle P, P \rangle = i(P)$ *for all non-zero* $P \in E_S(K)$.

Heights

A concrete example

If $\lambda : K^\times \backslash A_K^\times / \prod_{v \notin S} U_v \to \mathbb{Q}$ continuous homomorphism, consider the composition and I have

$$(E(K) \otimes \mathbb{Q}_p) \times (E(K) \otimes \mathbb{Q}_p) \to \mathbb{Q}_p \quad (P, Q) \mapsto \langle P, Q \rangle_\lambda,$$

with $\langle P, P \rangle_\lambda = \lambda(i(P))$ (logarithm: formula with sigma function).

Proposition

If $P \in E_S(Q)$, then $\langle P, P \rangle_\lambda = \log_p(\sigma_{p,\omega}^2(P)/d)$, where d is the denominator of the rational number $x(P)$ (as a fraction in lowest terms).

Technical preliminaries

Bad reduction primes

- K number field; E/K elliptic curve. Two kind of primes dividing p:
  1. Néron model split multiplicative at v. $q_v = q(j^{-1}(E)) \in K_v^\times$ multiplicative period $i_v : K_v^\times \to E(K_v)$.
  2. The other ones.
- $K_p = K \otimes \mathbb{Q}_p = \prod_{v|p} K_v$, $E(K_p) = \prod_{v|p} E(K_v)$. Write

$$E^\dagger(K_p) := \prod_{v \text{ of type I}} K_v^\times \times \prod_{v \text{ of type II}} E(K_v).$$

- $0 \to \mathbb{Z}^N \xrightarrow{\phi} E^\dagger(K_p) \xrightarrow{\psi} E(K_p) \to 0$, where N number of places type I. $\phi(a_v)$ entry q_v in the v-th place and entry 1 in the others. ψ: for v type I, i_v and for type II identity.

Technical remarks

Still the previous pairing

Recall exact sequence

$$0 \to \mathbb{Z}^N \to E^\dagger(K) \to E(K) \to 0,$$

so $E^\dagger(K)$ finitely generated of rank $r + N$.

- S set of places of K dividing p. E ordinary reduction at $v \in S$ and S contains all v of type I.
- Canonical splitting $E_S(K)$, that is, a mapping $E_S(K) \to E^\dagger(K)$ such that $\psi(\bar{P}) = P$: if v type I, $w_v(P) \in O_v^\times$ unique unit with $i_v(w_v(P))$ image of P in $E(K_v)$. Then, $\tilde{P}$ has these entries for type I and image of P in $E(K_v)$ for type II.
- We will fixe now $\lambda : K^\times \backslash A_K^\times / \prod_{v \notin S} U_v \to \mathbb{Q}_p$ continuous homomorphism, with local factor at v denoted λ_v.

Another pairing

Definition

Proposition

There is a unique bilinear symmetric pairing

$$(E^\dagger(K)\otimes\mathbb{Q}_p)\times(E^\dagger(K)\otimes\mathbb{Q}_p)\to\mathbb{Q}_p,\qquad [(P,Q)\mapsto\langle P,Q\rangle_\lambda^\dagger]$$

depending only upon λ *and such that,*

$$\langle\tilde{P},\tilde{Q}\rangle_\lambda^\dagger=\langle P,Q\rangle_\lambda \quad \text{for } P,Q\in E_S(K)$$

$$\langle a_v,P\rangle_\lambda^\dagger=\lambda_v(w_v(P))/\operatorname{ord}_v(q_v), \quad \text{for } v \text{ of type } I \text{ and } P\in E_S(K)$$

$$\langle a_v,a_{v'}\rangle_\lambda^\dagger=\begin{cases}\lambda_v(q_v)/\operatorname{ord}_v(q_v), & \text{if } v=v'\\ 0, & \text{if } v\neq v',\end{cases} \quad \text{for } v,v' \text{ of type } I.$$

Last definitions

All the ingredients prepared

- The λ-sparsity of E, $S_\lambda(E)$ is defined to be

$$S_\lambda(E) := \det \langle P_i, P_j \rangle_\lambda^\dagger / t^2 \in \mathbb{Q}_p,$$

where $P_1, \ldots, P_{r+N}$ is a maximal system of linearly independent points in $E^\dagger(K)$ and t index of the subgroup they generate in $E^\dagger(K)$.

- Suppose $\lambda_v(q_v)$ all non-zero for v of type I. The Schneider λ-height

$$E(K) \times E(K) \to \mathbb{Q}_p, \qquad (P, Q) \mapsto \langle P, Q \rangle_\lambda^{\mathrm{Sch}}$$

is defined to be

$$\langle P, Q \rangle_\lambda^{\mathrm{Sch}} := \langle \tilde{P}, \tilde{Q} \rangle_\lambda - \sum_{v \text{ of type I}} \frac{\lambda_v(w_v(P)) \cdot \lambda_v(w_v(Q))}{\lambda_v(q_v) \cdot \mathrm{ord}_v(q_v)}.$$

Sparsity

And the multiplicative case

$R_\lambda^{\mathrm{Sch}}(E)$ discriminant of the lattice $E(K)/\text{torsion}$ in $E(K) \otimes \mathbb{Q}_p$ with Schneider λ-height.

Proposition

Suppose that $\lambda_v(q_v)$ nonzero for all v of type I. Then,

$$S_\lambda(E) = \Big(\prod_{v \text{ of type } I} \mathcal{L}_\lambda(E_{/K_v}) \Big) R_\lambda^{\mathrm{Sch}}(E) \cdot |E(K)_{tors}|^{-2}.$$

- $L_p(E, \psi, s)$ exceptional zero at $s = 1$ if $e_p(\alpha, j, \psi) = 0$ for $j = 0$ (ifff $p || N$ - multiplicative reduction and $\alpha = \psi(p)$.
- $L_p(E, \psi, s)$ exceptional zero at $s = 1$ iff either E split multiplicative at p and $\psi(p) = 1$ or E non-split multiplicative and $\psi(p) = -1$.

Classical BSD

The conjecture

Let $E/\mathbb{Q}$ be an elliptic curve. The Hasse-Wil L-function is just the Mellin transform of the newform f related to E.
Let $L^{(k)}(E) := (1/k!) \cdot \frac{d^k}{ds^k} L(E,s)|_{s=1}$.

Conjecture (Classical BSD)

1. $L^{(k)}(E) = 0$ *for* $k < r$.
2. $L^{(r)}(E) = |W(E_{/\mathbb{Q}})| \cdot \frac{R_\infty(E_{/\mathbb{Q}})}{|E(\mathbb{Q})|_{\text{tors}}|^2} \left(\prod_l m_l\right) \Omega_E^+.$

Here $R_\infty(E)$ is the classical regulator.

p-adic BSD

Non-exceptional case

p prime of good, ordinary reduction for E or a prime such that the Néron fiber $E_{/\mathbb{F}_p}$ is multiplicative ($p||N$). α unique allowable p-root for f. For Dirichlet characters ψ, define

$$L_p(E, \psi, s) := L_p(f, \alpha, \psi, s-1)$$

and if $\psi = 1$ put $L_p(E, \psi, s) = L_p(E, s)$.
In the non-exceptional case, when $\alpha \neq 1$ (E has good ordinary or non-split multiplicative reduction at p), then

$$L_p^{(k)}(E) = 0 \text{ for } k < r, \text{ the rank of } E(\mathbb{Q}),$$

$$L_p^{(r)}(E) = \left(1 - \frac{1}{\alpha}\right)^b \cdot |W(E_{/\mathbb{Q}})| \cdot S_P(E_{/\mathbb{Q}}) \cdot \left(\prod_l m_l\right)\Omega_E^+,$$

where $b = 2$ when E has good reduction at p and 1 elsewhere.

p-adic BSD

Exceptional case

In the exceptional case, $\alpha = 1$ (split multiplicative reduction), then

$$L_p^{(k)}(E) = 0 \text{ for } k < r+1$$

$$L_p^{(r+1)}(E) = |W(E_{\mathbb{Q}})| \cdot S_p(E_{/\mathbb{Q}}) \cdot \Big(\prod_l m_l\Big)\Omega_E^+$$

$$= \mathcal{L}_p(E_{/\mathbb{Q}_p}) \cdot |W(E_{/\mathbb{Q}})| \frac{R_p^{\mathrm{Sch}}(E_{/\mathbb{Q}})}{|E(\mathbb{Q})_{\mathrm{tors}}|^2} \Big(\prod_l m_l\Big)\Omega_E^+.$$

In [MTT] they also state twisted conjectures.

Further work in this direction

See you along the week

- The article "The p-adic L-function of modular elliptic curves" of Bertolini and Darmon summarizes future work.
- The (unexpected) appearance of Tate's period q in the derivative of $L_p(E,s)$ led Schneider to see a purely p-adic analytic construction of $L_p(E,s)$ relying on a p-adic uniformization of E.
- Schneider's program does not recover the p-adic L-function of Mazur and Swinertonn-Dyer.
- Bertolini and Darmon (1996) laid the foundation for a parallel study in which we change the cyclotomic setting by the anticyclotomic one: the role of modular symbols is now played by Heegner points.