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Iwasawa theory and STNB, a personal view

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Cyclotomic extensions: number fields

For any $n \in \mathbb{N}$ and p an odd prime, let

- $\mathbb{Q}_n := \mathbb{Q}(\boldsymbol{\mu}_{p^{n+1}})$
$$\mathbb{Q} \subset \mathbb{Q}_0 \subset \mathbb{Q}_1 \subset \cdots \subset \mathbb{Q}_n \subset \cdots \subset \mathbb{Q}(\boldsymbol{\mu}_{p^\infty}) \ .$$
- $\mathbb{Q}_{cyc} := \bigcup \mathbb{Q}(\boldsymbol{\mu}_{p^n})$ the p-cyclotomic extension of $\mathbb{Q}$.
- $\mathbb{Q}_{cyc}/\mathbb{Q}_0$ is a $\mathbb{Z}_p$-extension: let $\mathrm{Gal}(\mathbb{Q}_{cyc}/\mathbb{Q}_0) := \Gamma$.

Properties

1. $\mathrm{Gal}(\mathbb{Q}_{cyc}/\mathbb{Q}) \simeq \varprojlim_n \left(\mathbb{Z}/p^{n+1}\right)^* \simeq \mathbb{Z}_p^* \simeq \mathbb{Z}/(p-1) \times \mathbb{Z}_p \simeq \mathrm{Gal}(\mathbb{Q}_0/\mathbb{Q}) \times \Gamma$.
2. ramified only at ∞ and p, in particular $\mathbb{Q}_{cyc}/\mathbb{Q}_0$ is ramified (totally) only at p.

$\mathbb{Q}_{cyc}^{\mathrm{Gal}(\mathbb{Q}_0/\mathbb{Q})}$ is called the **cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$**.
For any number field K, $K\mathbb{Q}_{cyc}^{\mathrm{Gal}(\mathbb{Q}_0/\mathbb{Q})}$ is the **cyclotomic $\mathbb{Z}_p$-extension of K**.
The **Iwasawa algebra** is the ring

$$\Lambda := \varprojlim_n \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}_0)] = \mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T]] \ .$$

The last isomorphism is non-canonical and given by $\gamma \mapsto T-1$ where γ is a chosen topological generator of Γ.

Carlitz Cyclotomic extensions: function fields

Let

- $F := \mathbb{F}_q(\theta)$ with $q = p^r \geqslant 3$ and fix $\frac{1}{\theta}$ as the prime at ∞.
- Let $A := \mathbb{F}_p[\theta]$ and fix a prime $\mathfrak{p}$ of A of degree d.
- Let Φ be the **Carlitz module** associated to A: it is an $\mathbb{F}_q$-linear ring homomorphism

  $$\Phi : A \to F\{\tau\}$$

  $$\theta \mapsto \Phi_\theta = \theta\tau^0 + \tau \ ,$$

  where $F\{\tau\}$ is the skew polynomial ring with $\tau f = f^q \tau$ for any $f \in F$.
- For any ideal $\mathfrak{a}$ of A write

  $$\Phi[\mathfrak{a}] := \{x \in \overline{F} \mid \Phi_a(x) = 0 \ \forall\, a \in \mathfrak{a}\} \ ,$$

  it is an A-module isomorphic to $A/\mathfrak{a}$.

Cyclotomic extensions: function fields

For any $n \in \mathbb{N}$, let

- $F_n := F(\Phi[\mathfrak{p}^{n+1}])$

$$F \subset F_0 \subset F_1 \subset \cdots \subset F_n \subset \cdots \subset F(\Phi[\mathfrak{p}^\infty]) \ .$$

- $\mathcal{F} := \bigcup F(\Phi[\mathfrak{p}^n])$ the $\mathfrak{p}$-cyclotomic extension of F.
- $\mathcal{F}/F_0$ is a $\mathbb{Z}_p^\infty$-extension: let $\mathrm{Gal}(\mathcal{F}/F_0) := \Gamma$.

Properties

1. $\mathrm{Gal}(\mathcal{F}/F) \simeq \varprojlim_n \left(A/\mathfrak{p}^{n+1}\right)^* \simeq \mathrm{Gal}(F_0/F) \times \Gamma := \Delta \times \Gamma$.
2. ramified only at ∞ and $\mathfrak{p}$, in particular $\mathcal{F}/F_0$ is ramified (totally) only at $\mathfrak{p}$ and the inertia group of ∞ is $\mathbb{F}_q^* \hookrightarrow \Delta$ (note $|\Delta| = q^{\deg(\mathfrak{p})} - 1 = q^d - 1$).

The **Iwasawa algebra** is the ring

$$\Lambda := \varprojlim_n \mathbb{Z}_p[\mathrm{Gal}(F_n/F_0)] = \mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T_n \ : \ n \in \mathbb{N}]].$$

Iwasawa modules: global fields

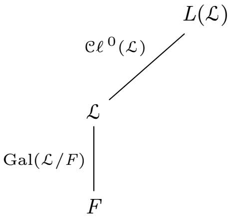
F a global field and E/F a finite extension

- $\mathcal{C}\ell^0(E)$ the p-part of the group of divisor classes of E of degree 0 (class group);
- $L(E)$ the maximal unramified abelian p-extension of E;
- $\mathcal{C}\ell^0(E) \simeq \mathrm{Gal}(L(E)/E)$ via the (canonical) Artin map.
- $\Lambda(\mathcal{L}) := \mathbb{Z}_p[[\mathrm{Gal}(\mathcal{L}/F)]]$ the associated Iwasawa algebra.

Same notations for infinite extensions $\mathcal{L}/F$ where

$$\mathcal{C}\ell^0(\mathcal{L}) := \varprojlim_{E} \mathcal{C}\ell^0(E)$$

(the limit is on the natural norm maps as E runs among the finite subextensions of $\mathcal{L}/F$).



$\mathcal{C}\ell^0(\mathcal{L})$ is a $\Lambda(\mathcal{L})$-module (the action is provided by conjugation)

Iwasawa modules: global fields

Theorem (Iwasawa 60's, Greenberg 70's,...)

Let $\mathcal{L}/F$ be a $\mathbb{Z}_p^d$-extension ($d < \infty$), (which is ramified in a finite set of places) then $\mathcal{C}\ell^0(\mathcal{L})$ is a finitely generated torsion $\Lambda(\mathcal{L})$-module.

Theorem (Structure Theorem for f.g.t. Iwasawa modules)

Let $\mathcal{L}/F$ be a $\mathbb{Z}_p^d$-extension ($d < \infty$) and let M be a finitely generated torsion $\Lambda(\mathcal{L})$-module. There is a pseudo-isomorphism (i.e. with pseudo-null kernel and cokernel)

$$M \sim_{\Lambda(\mathcal{L})} \bigoplus_{i=1}^{s} \Lambda(\mathcal{L})/(f_i^{e_i})$$

where the f_i are irreducible elements of $\Lambda(\mathcal{L}) \simeq \mathbb{Z}_p[[T_1, \ldots, T_d]]$. A f.g.t. module N is pseudo-null *if $ht(Ann_{\Lambda(\mathcal{L})}(N)) \geqslant 2$.*

Iwasawa modules: characteristic ideals

Definition (Characteristic ideal)

For a f.g.t module as above we define the characteristic ideal *as*

$$Ch_{\Lambda(\mathcal{L})}(M) := \left(\prod_{i=1}^{s} f_i^{e_i}\right) .$$

A f.g.t. module N is pseudo-null *(i.e.* $N \sim_{\Lambda(\mathcal{L})} 0$*)* $\iff$ $Ch_{\Lambda(\mathcal{L})}(N) = (1)$.

Theorem (Iwasawa)

Let K_∞/K *be a* $\mathbb{Z}_p$*-extension of a number field* K *and let* $\mathcal{C}\ell^0(K_n)$ *be the* p*-part of the class group of the* n*-th layer of* K_∞*. Then there exist nonnegative integers* μ, λ *and* ν *such that*

$$|\mathcal{C}\ell^0(K_n)| = p^{\mu p^n + \lambda n + \nu} \quad \forall n \gg 0 .$$

Let $\mathcal{C}\ell^0(K_\infty)$ be the Iwasawa module associated to K_∞ and let f_{K_∞} be a (polynomial) generator of $Ch_{\Lambda(K_\infty)}(\mathcal{C}\ell^0(K_\infty))$. Then

- $p^\mu \mid f_{K_\infty}$ and $p^{\mu+1} \nmid f_{K_\infty}$;
- $\lambda = \deg(f_{K_\infty})$.

Main Conjecture: number fields

Let χ be a Dirichlet character associated to the field K, then

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}\ Re(s)>1.$$

For any character ω^i (i even and nonzero and ω the Teichmüller character) associated to $\mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_p)/\mathbb{Q})$ Iwasawa defined p-adic analogues $L_p(s,\omega^i)$ of $L(s,\omega^i)$.

$$L_p(1-m,\omega^i)=(1-\omega^{i-m}(p)p^{m-1})L(1-m,\omega^{i-m})\ for\ m\geq 1$$

Moreover Iwasawa proved that there exist a power series $f(T,\omega^i)\in\mathbb{Z}_p[[T]]\simeq\Lambda(\mathbb{Q}_{cyc})$ such that $L_p(s,\omega^i)=f((1+p)^s-1,\omega^i)$.

Conjecture (Main Conjecture (MC))

Let $\mathcal{C}\ell^0(\mathbb{Q}_{cyc})(\omega^i)$ *be the* ω^i*-part of the Iwasawa module, then*

$$Ch_{\Lambda(\mathbb{Q}_{cyc})}(\mathcal{C}\ell^0(\mathbb{Q}_{cyc})(\omega^i))=(f(T,\omega^{1-i}))\ .$$

First proof by Mazur-Wiles *Invent. Math. '84*. Different proofs and many generalizations have been investigated since then.

Fitting-Char-Determinant

- $D^b(R)$ the derived category of bounded complexes of finitely generated left projective R-modules, (in particular complexes $C^\bullet$ such that $H^i(C^\bullet) = 0$ except for a finite list of i, recall: acyclic if $H^i(C^\bullet) = 0\ \forall i$).
  $C^\bullet$ a complex of R-f.g. modules is named perfect if it is quasi-isomorphic to a bounded complex $C'^\bullet$ with $C'^i = 0$ for $\forall |i| \geq n(C')$ and C'^j is projective f.g.R-module.
- A denotes a commutative ring with 1.
  M a finite presentation A-module (in particular finite generated).

Definition (Fitting ideal)

Let $A^a \xrightarrow{\phi} A^b \twoheadrightarrow M$ *be a finite presentation for an* A*-module* M*, then*

$$\mathrm{Fitt}_A(M) = \begin{cases} 0 & \text{if } a < b \\ \text{the ideal generated by all the determinants of the } b \times b \text{ minors of the matrix } \phi & \text{if } a \geqslant b \end{cases}.$$

- For A-non-noetherian there is a definition of *Fitt* for f.g. A-modules.

Fitting-Char-Determinant

- If A semilocal noetherian, $C^{\bullet}$ acyclic outside degree i and $H^i(C^{\bullet})$ is a f.g. torsion A-module of $pd_A(H^i(C^{\bullet}) \leq 1$, then $Quot(A) \otimes^{\mathbb{L}}_A C^{\bullet}$ acyclic, and $Fit_A(H^i(C^{\bullet}))$ invertible ideal of A.
- If A is Iwasawa algebra $\mathbb{Z}_p[\Delta][[T_1, \dots, T_n]]$ and M is torsion A-module and $pd_A(M) \leq 1$ then $ch_A(M) = Fit_A(M)$.
- Knudsen-Mumford determinant, (A-noetherian): $Det : D^b(A) \to Picard(A)$, for this talk, restrict $C^{\bullet}$ acyclic outside degree i, and $H^i(C^{\bullet})$ is a f.g. torsion A-module of $pd_A(H^i(C^{\bullet})) \leq 1$, then:

$$Det(C^{\bullet}) = (Fit_A(H^i(C^{\bullet}))^{(-1)^{i+1}}).$$

Non-commutative generalizations for Det:

- Burns-Flach: Extends to non-commutative Iwasawa algebras Det-functor by virtual objects (Deligne).
- Kato, Burns, Coates, Venjakob, Schneider, Fukaya, ...: Extends to non-commutative Iwasawa rings by use of K-theory.
- Witte: The interpretation of K-theory can be understood inside K-theory of Waldhausen categories.

Iwasawa algebras for positive char.

For simplicity $k = \mathbb{F}_q(T)$, once and for all.
Denote $\mathbb{F}_\infty$ where $\mathbb{F}_\infty/\mathbb{F}_q$ is the unique $\mathbb{Z}_p$-extension of $\mathbb{F}_q$.
Consider now K/k, K the rational field of a curve C_K, with $C_K \to C_k = \mathbb{P}^1_{\mathbb{F}_q}$ Galois cover such that $Gal(K/k)$ is a profinite group and its p-Sylow subgroup has finite index, and K/k is ramified in a finite set of places of k.
Σ: a finite set of places of k containing the ramified places of K/k.

Interesting Galois covers for Iwasawa theory:

1. Burns-Kato-Witte-...: Assume moreover $Gal(K/k)$ is topologically finitely generated (i.e. p-adic Lie group).
   Appears the notion of Λ-ring (Fukaya-Kato): if exists two-sided ideal I of Λ such that Λ/I^n finite and $\Lambda \cong \varprojlim \Lambda/I^n$.
   Fact: Λ adic $\mathbb{Z}_p$-algebra and G profinite group which is topologically generated and has an p-Sylow subgroup of finite index, then $\Lambda[[G]]$ is an adic ring.
2. We wish: $Gal(K/k)$ not necessarily topologically finitely generated in order to consider Carlitz cyclotomic powers or in non-commutative setting torsion of higher Drinfeld modules (for example $GL_2(\mathbb{F}_q[[T]])$-extensions, following Pink school).

In any case consider the Iwasawa algebra $\Lambda(Gal(K/k)) := \mathbb{Z}_p[[Gal(K/k)]]$.

A module on geometric Iwasawa algebras

Denote by

$$X_{K/k} := \varprojlim_{U} \mathbb{Z}_p \otimes Cl^0(K^U)$$

where U over open subgroups of $Gal(K/k)$, a $\Lambda(Gal(K/k))$-module.
The cohomological interpretation of $X_{K/k}$ is explained by exact sequences:

$$0 \to H^0(D^\bullet_{K/k}) \to \oplus_{v \in \Sigma^K_{fin}} \Lambda(Gal(K/k)) \otimes_{\Lambda(Gal(K/k)_v)} \mathbb{Z}_p \to X_{K/k} \to$$

$$\varprojlim_{Norm} \mathbb{Z}_p \otimes Y_{K/k} := \varprojlim_{U} Cl(\mathcal{O}_{K^U,\Sigma}) \to 0$$

$$0 \to \varprojlim_{Norm} \mathbb{Z}_p \otimes Y_{K/k} \to H^1(D^\bullet_{K/k}) \to \oplus_{v \in \Sigma} \Lambda(Gal(K/k)) \otimes_{\Lambda(Gal(K/k)_v)} \mathbb{Z}_p \to \mathbb{Z}_p \to 0$$

Where

$$D^\bullet_{K/k} := RHom_{\Lambda(Gal(K/k))}(R\Gamma_{et}(C_k, j_{k,\Sigma !}(\Lambda(Gal(K/k)^{\#}), \Lambda(Gal(K/k))[-2]).$$

and $j_{k,\Sigma} : Spec(\mathcal{O}_{k,\Sigma}) \to C_k$.

Theorem (Burns)

Suppose $Gal(K/k)$ is p-adic Lie Group.

1. *If for each $v \in \Sigma$, $Gal(K/k)_v$ is non-trivial and normal, OR*
2. *If $\mathbb{F}_{q^{p^\infty}} \subset K$,*

Then, $D^\bullet_{K/k,\Sigma}$ is acyclic outside degree 1 and $H^1(D^\bullet_{K/k,\Sigma})$ is $\Lambda(Gal(K/k))$-finitely generated torsion module with $pd_{\Lambda(Gal(K/k))}(H^1(D^\bullet_{K/k,\Sigma})) \leq 1$.

Theorem (Anglès-Bandini-B.-Longhi)

Consider K/k the Carlitz cyclotomic extension. Then $(W \otimes_{\mathbb{Z}_p} X_{K/k})(\chi)$ is a finitely generated torsion $W \otimes_{\mathbb{Z}_p} \Lambda(Gal(K/k))(\chi)$-module, where χ any non-trivial p-adic character of Δ in W the Witt ring of $\mathbb{F}_{\mathfrak{p}}$.

GIMC: abelian case

$\mathscr{P}_k$ the set of places of k.
Define G_Σ as the Galois group of the maximal abelian extension $\mathcal{F}_\Sigma$ of k which is unramified outside Σ. For any $\mathfrak{q} \in \mathscr{P}_F \setminus \Sigma$, let $\mathrm{Fr}_\mathfrak{q} \in G_\Sigma$ denote the corresponding (arithmetic) Frobenius automorphism.

Definition

We define the Stickelberger series *by*

$$\Theta_{\mathcal{F}_S/F,\Sigma}(X) := \prod_{\mathfrak{q}\in\mathscr{P}_F-\Sigma} (1-\mathrm{Fr}_\mathfrak{q}^{-1}X^{\deg(\mathfrak{q})})^{-1} \in \mathbb{Z}[[G_\Sigma]][[X]] \ . \qquad (1)$$

More generally, for any closed subgroup $U < G_\Sigma$, we define

$$\begin{aligned}\Theta_{\mathcal{F}_S^U/F,S}(X) &:= \pi^{G_S}_{G_S/U}(\Theta_{\mathcal{F}_S/F,S})(X) \\ &= \prod_{\mathfrak{q}\in\mathscr{P}_k-\Sigma} (1-\pi^{G_\Sigma}_{G_\Sigma/U}(\mathrm{Fr}_\mathfrak{q}^{-1})X^{\deg(\mathfrak{q})})^{-1} \in \mathbb{Z}[\mathrm{Gal}(\mathcal{F}_\Sigma^U/k)][[X]] \ ,\end{aligned}$$

where $\pi^{G_\Sigma}_{G_\Sigma/U} : \mathbb{Z}[[G_\Sigma]] \to \mathbb{Z}[[\mathrm{Gal}(\mathcal{F}_\Sigma^U/k)]]$ is the map induced by the projection $G_S \twoheadrightarrow G_\Sigma/U$.

Theorem (Burns,(Crew, Burns-Lai-Tan))

K/k abelian p-adic Lie extension, and no place of Σ splits completely in K/k, then

$$Det(D^{\bullet}_{K/k})^{-1} = Fitt_{\Lambda(Gal(K/k))}(H^1(D^{\bullet}_{K/k})) = (\Theta_{K/k,\Sigma}(1)).$$

Theorem (Anglès-Bandini-B.-Longhi)

K/k Carlitz cyclotomic extension, then

$$Fitt_{\Lambda(Gal(K/k)(\chi)\otimes_{\mathbb{Z}_p} W}(W \otimes_{\mathbb{Z}_p} X_{K/k}(\chi)) = (e_{\chi}(\Theta_{K/k,\{\infty,\mathfrak{p}\}}(1))),$$

if χ an odd p-adic character, W is the Witt ring of $\mathbb{F}_{\mathfrak{p}}$. (For even χ non-trivial we obtain also a statement).

Where here $\chi = \tilde{\omega}_{\mathfrak{p}}^{i}$ (a power of the Teichmüller character in zero characteristic), and even if and only if $q-1$ divides i, otherwise odd.

Link of IMC with positive characteristic Goss-(Pink-Boeckle) L-values

Let $A_{\mathfrak{p}}$ be the completion of $A = \mathbb{F}_q[T]$ at $\mathfrak{p}$ and $Dir(\mathbb{Z}_p, A_{\mathfrak{p}})$ the $A_{\mathfrak{p}}$-module of *Dirichlet series*, i.e., the closure in $C^0(\mathbb{Z}_p, A_{\mathfrak{p}})$ of the module generated by functions $\vartheta_u(y) := y^u$, $u \in U_1 = 1 + \mathfrak{p}A_{\mathfrak{p}} \cong \Gamma$.
We shall use a result of Sinnot (2008) to obtain a map

$$s_X : W[[Gal(K/k)]][[X]] \longrightarrow Dir(\mathbb{Z}_p, \mathbb{F}_q[T]_{\mathfrak{p}})[[X]] \ ,$$

and prove

Theorem (Anglès-Bandini-B-Longhi)

For every $y \in \mathbb{Z}_p$ and $i \in \mathbb{Z}/(q^d - 1)$ we have

$$s_X(\theta_{K/k,\{\mathfrak{p},\infty\}}(X, \tilde{\omega}_{\mathfrak{p}}^{-i}))(y) = L_{\mathfrak{p}}(X, -y, \omega_{\mathfrak{p}}^i) \in \mathbb{F}_q[T]_{\mathfrak{p}}[[X]],$$

where $L_{\mathfrak{p}}$ is a $\mathfrak{p}$-adic L-function.

We mention here for $j \equiv i(mod q^d - 1)$ $L_{\mathfrak{p}}(X, j, \omega_{\mathfrak{p}}^i) = (1 - \pi_{\mathfrak{p}}^j X^d)Z(X, j)$,
where $Z(X, j) := \sum_{n \geq 0} S_n(j)X^n \in A[X]$, with $S_n(j) := \sum_{a \in A_{+,n}} a^j$.
and for $j \not\equiv 0(mod\ q - 1)$ $(1 - \pi_{\mathfrak{p}}^j)\beta(j) = L_{\mathfrak{p}}(1, j, \omega_{\mathfrak{p}}^i) \neq 0$, the Bernouilli-Goss numbers $\beta(j) = Z(1, j)$.
And have an Ferrero-Washington style result:

Theorem (Anglès-Bandini-B-Longhi:"μ vanishes")

For $i \not\equiv 0(mod\ q - 1)$ then $\theta_{K/k,\{\mathfrak{p},\infty\}}(1, \tilde{\omega}_{\mathfrak{p}}^i) \not\equiv 0(mod\ p)$

From commutative to non-commutative

For the classical cyclotomic extension over $\mathbb{Q}$, $G_{cyc} := Gal(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \cong \Delta \times \mathbb{Z}_p$ and $\Lambda_{cyc} \cong \mathbb{Z}_p[\Delta][[T]]$, consider:

$$S = \{s \in \Lambda_{cyc} | \Lambda_{cyc}/s\Lambda_{cyc}\ f.g.\mathbb{Z}_p[\Delta]-mod\}$$

and the localization sequence in K-theory:

$$K_1(\Lambda_{cyc}) \to K_1(\Lambda_{cyc,S}) \to^d K_0(\Lambda_{cyc}, \Lambda_{cyc,S}) = K_0(\mathfrak{M}_\Delta(\Lambda_{cyc}))$$

where $\mathfrak{M}_\Delta(\Lambda_{cyc})$ f.g. Λ_{cyc}-modules s.t. f.g. $\mathbb{Z}_p[\Delta]$-modules(=torsion Λ_{cyc}-modules).

- $d(s) = [\Lambda_{cyc}/s\Lambda_{cyc}]$, $s \in S$.
- $(f) = (Fitt_{\Lambda_{cyc}}(M))$, $d(f) = [M]$.

Localization sequence for rings

Definition

R ring $S \subset R$ is a left denominator set if is multiplicatively closed and satisfies:

- *Ore condition: for each $s \in S, b \in R$ exists $s' \in S, b' \in R$ such that $b's = s'b$.*
- *annihilator condition: for each $s \in S, b \in R$ with $bs = 0$ exist $s' \in S$ with $s'b = 0$.*

We have localization sequence:

$$K_1(R) \to K_1(S^{-1}R = R_S) \to^d K_0(R, R_S)$$

Extending localization to complexes

Theorem (Weibel-Yao,Muro,Muro-Tonks,Witte)

- *The group $K_0(R, R_S)$: generators $[P^\bullet]$ for $P^\bullet$ perfect complex of R-modules, s.t. localization $SP^\bullet$ is an acyclic complex.*
  Relations: not in the talk.
- *The abelian group $K_1(R)$: generators $[f]$ where f is a quasi-automorphism of a perfect complex of R-modules $P^\bullet$. (Relations:not in the talk)*
- *$K_1(R_S)$: generators $[f]$ with f a morhism of perfect complex $P^\bullet$ such that f_S is quasi-automorphism.*
- *$d : K_1(R_S) \to K_0(R, R_S)$ given by $[f] \mapsto -[Cone(f)^\bullet]$.*

Localization in non-commutative Iwasawa algebras

- The cover K/k for general schemes over $\mathbb{F}_q$ instead of curves:
  The p-adic Lie extension $Y \to X$ (Galois pro-finite cover factors through $X \times_{\mathbb{F}_q} \mathbb{F}_{q^{p^\infty}}$ with p-Sylow of finite index with $Gal(Y/K)$ and admits a finite set of topological generators) we think now X separated and geometrically connected scheme of finite type over $\mathbb{F}_q$.
  $H := Ker(G := Gal(Y/X) \to Gal(\mathbb{F}_{q^{p^\infty}}/\mathbb{F}_q))$, $Gal(Y/X) \cong H \rtimes Gal(\mathbb{F}_{q^{p^\infty}}/\mathbb{F}_q)$
- $S = \{s \in \Lambda(Gal(Y/X)) | \Lambda(G)/\Lambda(G)s, \ is\ f.g.\ \Lambda(H) - module\}$

Theorem (Venjakob)

Given M a f.g. $\Lambda(G)$-module, Then:
S-torsion if and only if M is f.g. $\Lambda(H)$-module.
$K_0(\Lambda(G), \Lambda(G)_S) = K_0(\mathfrak{M}_H(G))$.

Question

For a non-finite set of topological generators for G (of the cover $Y \to X$): given M as above and f.g. $\Lambda(H)$-module, is then M S-torsion?

Up to know G IS a p-adic Lie extension from above pro-finite cover $Y \to X$, and X proper over $\mathbb{F}_q$.

First part of geometric non-commutative IMC

$\mathcal{F}$ flat $\mathbb{Z}_p$-sheaf, U open subgroup of G, $f_U : Y^U \to X$, $(f : Y \to X)$, we have

Theorem (Deligne)

If $\mathcal{F}_n$ flat sheaf of $\mathbb{Z}/p^n$-modules, then $R\Gamma_{et}(X, f_{U,*}f_U^*\mathcal{F}_n)$ is a perfect complex of $\mathbb{Z}/p^n[G/U]$-modules.

Theorem (Burns,Witte)

$\mathcal{F}$ flat $\mathbb{Z}_p$-sheaf, then $R\Gamma_{et}(X, f_*f^*\mathcal{F})$ is a perfect complex of $\Lambda(G)$-modules.

First part of IMC for non-commutative Iwasawa rings

Theorem (Burns)

The $\Lambda(G)$-complex $R\Gamma_{et}(X, f_*f^*\mathcal{F})$ satisfies that $S^{-1}\Lambda(G) \otimes^{\mathbb{L}}_{\Lambda(G)} R\Gamma_{et}(X, f_*f^*\mathcal{F})$ is acyclic, and exists $\zeta(\mathcal{F}) \in K_1(\Lambda(G)_S)$ such that $d(\zeta(\mathcal{F})) = [R\Gamma_{et}(X, f_*f^*\mathcal{F})]$ in $K_0(\Lambda(G), \Lambda(G)_S)$.

Remark

Witte gave an unified treatment for ℓ-adic Lie extensions with $\ell \neq p$ and $\mathcal{F}$ a flat $\mathbb{Z}_\ell$-adic sheaf.

If X only separated, use compact étale complexes.

A complex generalization

Any Λ adic ring, and left denominator set S we have,

$$K_1(PDG^{cont}(\Lambda)) \to K_1(\Lambda_S) \to^d K_0(PDG^{cont}(\Lambda, S))$$

- PDG^{cont} is a category of inverse systems of complexes of left Λ-modules indexed by $\mathfrak{I}_\Lambda$ (a topological basis for the profinite cover, or the powers of the bilater ideal for a Λ adic ring), $(P_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ i.e.
  - for each $I \in \mathfrak{I}_\Lambda$, $P_I^\bullet$ a perfect complex of left Λ/I-modules (with further properties)
  - for $I \subset J \in \mathfrak{I}_\Lambda$, transition maps $\varphi_{IJ} : P_I^\bullet \to P_J^\bullet$ induces isomorphism

$$\Lambda/J \otimes_{\Lambda/I} P_I^\bullet \cong P_J^\bullet.$$

- $PDG^{cont}(\Lambda, S)$ objects of $PDG^{cont}(\Lambda)$ for which

$$0 \to S^{-1}\Lambda \otimes_\Lambda \varprojlim_{I\in\mathfrak{I}_\Lambda} P_I^\bullet$$

  is a quasi-isomorphism.

The generalization in the cover, $\ell \neq p$

For $f: Y \to X$ ℓ-adic Lie pro-cover, $\Lambda = \mathbb{Z}_\ell[[G]]$-ring of the cover, $\mathcal{F}_G := f_* f^* \mathcal{F}$ with $\mathcal{F}$ flat $\mathbb{Z}_\ell$-sheaf, and before we observed that

$$R\Gamma_{et,c}(X, f_* f^* \mathcal{F}) \in PDG^{cont}(\Lambda).$$

This notion can be extended to complex of sheaves and the above sequence in compact-étale to obtain:

Theorem (Witte, 2008)

Λ adic ring, such that the characteristic p is invertible in Λ. Assume $S \subset \Lambda$ is a left denominator set, and $R\Gamma_{et,c}(X, \mathcal{F}^\bullet_G) \in PDG^{cont}(\Lambda, S)$. Then

$$dL(\mathcal{F}^\bullet_G, 1) = [R\Gamma_{et,c}(X, \mathcal{F}^\bullet_G)] \in K_0(PDG^{cont}(\Lambda, S),$$

where

$$L(\mathcal{F}^\bullet_G, 1) = [S^{-1}\Lambda \otimes^{\mathbb{L}}_\Lambda R\Gamma_{c,et}(\overline{X}, \mathcal{F}^\bullet_G) \to^{id - Frob_{\mathbb{F}_q}} S^{-1}\Lambda \otimes^{\mathbb{L}}_\Lambda R\Gamma_{c,et}(\overline{X}, \mathcal{F}^\bullet_G)]$$

The source of Witte result

Take the particular case where $\Lambda = \mathbb{Z}_\ell$, and we consider only sheaves at place 0 of $\mathcal{F}^\bullet_G$, which are the $\mathbb{Z}_\ell$-sheaves over X, $\mathcal{F}$, (the cover $id : X \to X$)
Consider the complex $R\Gamma_{c,et}(X, \mathcal{F})$, and $S = \mathbb{Z}_\ell - \{0\} \subset \mathbb{Z}_\ell$.
By SGA5 $\zeta_{X,\mathcal{F}}(1) = \underline{det}(L(\mathcal{F}, 1)) = \prod_i det(1 - Frob_{\mathbb{F}_q} | H^i_{c,et}(\overline{X}, F \otimes \mathbb{Q}_\ell))^{(-1)^{i+1}}$, where $\underline{det} : K_1(A) \to A^*$ for a commutative ring A.
In particular $\zeta_{X,\mathcal{F}}(n) = L(\mathcal{F}(n), 1) = \prod_i det(1 - q^{-n} Frob_{\mathbb{F}_q} | H^i_{c,et}(\overline{X}, F \otimes \mathbb{Q}_\ell))^{(-1)^{i+1}}$
Consider n integer such that q^n is not eigenvalue of $Fr_{\mathbb{F}_q} | H^i_{c,et}(\overline{X}, \mathcal{F}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, $\forall i$ then [Bayer-Neukirch] proved that $H^i_{c,et}(X, \mathcal{F}(n))$ are finite, i.e. $R\Gamma_{c,et}(X, \mathcal{F}(n))$ is in $PDG^{cont}(\mathbb{Z}_\ell, S)$ (is S-acyclic, $\mathbb{Q}_\ell \otimes R_{c,et}(X, \mathcal{F}(n))$ is quasi-isomorphic to the complex of $0^\bullet$).
The exact sequence $K_1(\mathbb{Z}_\ell) \to K_1(\mathbb{Q}_\ell) \to K_0(PDG^{cont}(\mathbb{Z}_\ell, S))$ reads through determinant ($\underline{det}$) in K_1 and $\chi : K_0 \to \ell^{\mathbb{Z}}$ via $\chi(M^\bullet) = \prod_{n\in\mathbb{Z}} (H^n(M^\bullet))^{(-1)^n}$ to

$$0 \to \mathbb{Z}_\ell^* \to \mathbb{Q}_\ell^* \to^{x \mapsto |x|_\ell} \ell^{\mathbb{Z}} \to 0$$

Thus the SOURCE result of Witte corresponds to:

Theorem (Bayer-Neukirch, 1978)

Assume $p \neq \ell$. Consider n integer such that q^n is not eigenvalue of $Fr_{\mathbb{F}_q} | H^i_{c,et}(\overline{X}, \mathcal{F}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, $\forall i$ then,

$$|(\zeta_{X,\mathcal{F}}(n))|_\ell = \chi(R\Gamma_{c,et}(X, \mathcal{F}(n))) = \prod_i \# H^i_{c,et}(X, \mathcal{F}(n))^{(-1)^{i+1}}.$$

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