Computing Bianchi modular forms with character

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- Some work already done is by Cremona, Bygott, Lingham, Sengün, Gunnells, Yasaki.
- In the totally real case a lot of progress has been achieved. In fact:

Theorem (Dembélé, Donnelly, Greenberg, Voight)

There exists an algorithm that, given a totally real field F, a nonzero ideal $\mathfrak{N}$ of the ring of integers of F, and a weight $k \in (\mathbb{Z}_{\geq 2})^{[F:\mathbb{Q}]}$, computes the space $S_k(\mathfrak{N})$ of Hilbert cusp forms of weight k and level $\mathfrak{N}$ over F as a Hecke module.

- We aim to have a similar theorem for the Bianchi case.

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- We have an explicit description of the action of the Hecke operators on the space of modular symbols.
- The Manin trick (the Euclidean algorithm) yields an algorithm for writing an arbitrary modular symbol as a $\mathbb{Z}$-linear combination of a finite set of generating symbols, thereby recovering $S_2(N)$ as a Hecke module.

Our setting

- Let $K = \mathbb{Q}(\sqrt{-d})$, $d \in \mathbb{Z}_{>0}$, let $\mathcal{O}_K$ the ring of integers of K, and let

$$\mathcal{H}_3 := \mathbb{C} \times \mathbb{R}_{>0} \simeq \mathbb{R} \times i\mathbb{R} \times j\mathbb{R}_{>0} \subseteq \mathbb{H}.$$

- The group $\Gamma := \mathrm{PGL}_2(\mathcal{O}_K)$ acts on $\mathcal{H}_3$ (as subgroup of $\mathbb{H}$) by hyperbolic isometries.
- The set of cusp forms are the Γ-orbits of $\mathbb{P}^1(\mathcal{O}_K)$. The group Γ also acts on $\mathbb{P}^1(\mathcal{O}_K)$ as fractional linar transformations. Let $\mathcal{H}_3^* := \mathcal{H}_3 \cup \mathbb{P}^1(\mathcal{O}_K)$.
- Let $h_K := \#\,\mathrm{Cl}(K)$. For each $(a\colon b) \in \mathbb{P}^1(\mathcal{O}_K)$ we correspond the class spanned by the ideal $(a, b) \in \mathrm{Cl}(K)$.

Theorem

There is a 1 to 1 correspondence

$$\mathrm{Cl}(K) \quad \longleftrightarrow \quad \{\Gamma\text{-orbits of } \mathbb{P}^1(\mathcal{O}_K)\}$$

Distance function and fundamental domain

We start by defining a *distance to cusps* function. Let $(z, t) \in \mathcal{H}_3$ and let $\alpha = (a\colon b) \in \mathbb{P}^1(\mathcal{O}_K)$. Then:

$$\delta((z,t),\alpha) := \frac{N_{K/\mathbb{Q}}((a,b))^2 t}{N_{K/\mathbb{Q}}(-bz+a)^2}$$

It is easy to check that this function is invariant under the action of $\mathrm{PGL}_2(\mathcal{O}_K)$.

For any $\alpha \in \mathbb{P}^1(\mathbb{Q})$, we define

$$H_\alpha := \{(z,t) \in \mathcal{H}_3 \colon \delta((z,t),\alpha) \leq \delta((z,t),\beta),\ \forall\ \beta \in \mathbb{P}^1(\mathcal{O}_K)\}.$$

By invariance above, we have that $\forall g \in \Gamma$, $gH_\alpha = H_{g\alpha}$.

Let $\alpha_1, \ldots, \alpha_{h_K}$ be a set of representatives of Γ-orbits of $\mathbb{P}^1(\mathcal{O}_K)$. Let

$$\mathcal{F} := H_{\alpha_1} \cup \cdots \cup H_{\alpha_{h_K}}$$

One can always choose the set of α_i such that $\mathcal{F}$ is connected.

Adding the level

Let $\mathfrak{n}$ be a prime ideal of $\mathcal{O}_K$. Then define

$$\Gamma(\mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \colon c \in \mathfrak{n} \right\}$$

One can prove that $[\Gamma \colon \Gamma \cap \Gamma(\mathfrak{n})] < \infty$ so computing a domain for $\Gamma(\mathfrak{n})$ is to translate the domain for Γ by the elements of the quotient.

Hecke operators I

Let $\Gamma_{\mathfrak{a}} := \mathrm{GL}(\mathcal{O}_K \oplus \mathfrak{a}) = \left\{ \begin{pmatrix} \mathcal{O}_K & \mathfrak{a}^{-1} \\ \mathfrak{a} & \mathcal{O}_K \end{pmatrix} \right\} \cap \mathrm{GL}(K)$. We have a decomposition of $\Gamma \backslash \mathcal{H}_3$ as:

$$\Gamma \backslash \mathcal{H}_3(\mathbb{C}) = \bigsqcup_{[\mathfrak{a}] \in \mathrm{Cl}(K)} \Gamma_{\mathfrak{a}} \backslash \mathcal{H}_3.$$

By a generalisation of Eichler-Shimura,

$$S_2(\mathfrak{n}) \cong \bigoplus_{[\mathfrak{a}] \in \mathrm{Cl}(K)} H^1(\Gamma_{\mathfrak{a}}, \mathbb{C})^+.$$

Let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ that does not divide $\mathfrak{n}$ nor the discriminant of K. We define the Hecke operator $T_{\mathfrak{p}}$ for a given element of $\S_2(\mathfrak{n})$ component wise as follows:

Assume that $f \in H^1(\Gamma_{\mathfrak{b}}, \mathbb{C})$ for some $[\mathfrak{b}] \in \mathrm{Cl}(K)$. Then, $T_{\mathfrak{p}} f \in H^1(\Gamma_{\mathfrak{a}}, \mathbb{C})$, satisfying $[\mathfrak{b}] = [\mathfrak{p}^{-1}\mathfrak{a}]$. For any fractional ideal $\mathfrak{c}$, let

$$I_{\mathfrak{c}} := \left\{ \begin{pmatrix} \mathcal{O}_K & \mathcal{O}_K \\ \mathfrak{c} & \mathfrak{c} \end{pmatrix} \right\} \cap \mathrm{GL}_2(K).$$

Hecke Operators II

Now let

$$\Theta(\mathfrak{p})_{\mathfrak{a},\mathfrak{b}} := \Gamma_{\mathfrak{b}} \backslash \{\varpi \in I_{\mathfrak{b}} I_{\mathfrak{a}}^{-1} \colon \det(\varpi)\mathfrak{a} = \mathfrak{p}\mathfrak{b}\}.$$

Let $\gamma \in \Gamma_{\mathfrak{a}}$ s.t. $\gamma I_{\mathfrak{a}} = I_{\mathfrak{a}}$. Then the map $\varpi \mapsto \varpi\gamma$ on $\mathrm{GL}_2(K)$ induces a bijection of the equivalence classes of $\Theta(\mathfrak{p})_{\mathfrak{a},\mathfrak{b}}$. Therefore, for every $\varpi \in \Theta(\mathfrak{p})_{\mathfrak{a},\mathfrak{b}}$, there exists $\delta_{\varpi} \in \Gamma_{\mathfrak{a}}$ and $\varpi_{\gamma} \in \Theta(\mathfrak{p})_{\mathfrak{a},\mathfrak{b}}$ such that

$$\varpi\gamma = \delta_{\varpi}\varpi_{\gamma}.$$

Finally, for any $z \in \mathcal{H}_3$, one has

$$(T_{\mathfrak{p}} f)(z) = \sum_{\varpi \in \Theta(\mathfrak{p})_{\mathfrak{a},\mathfrak{b}}} f(\delta_{\varpi})^{\varpi}.$$

Implementation. Possible issues. Closing thoughts

- We are constructing the fundamental domain with a different philosophy than previous authors, who computed it directly in the Poincaré disc model of $\mathcal{H}_3$. We expect numerical accuracy from following this more classical approach. It will also be an improvement on the currant Magma implementation, since it uses Voronoi tesselation due to Yasaki and Gunnells.

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- We can also precompute generators for $\mathrm{Cl}(K)$, which will speed up the splitting process.
- We are still finishing the details on the Hecke action! Some details from last slide might need fine tuning.
- In the end, we expect you to be able to feed the program an imaginary quadratic field $K/\mathbb{Q}$, a level $\mathfrak{n}$ and a character χ and receive the space $S_2(\mathfrak{n}, \chi)$, for which you would be able to compute the $T_n(f)$'s for any n.