A second derivative result for p-adic L-functions (after Bertolini–Darmon)

Carlos de Vera Piquero (Essen)

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Setting

- $E/\mathbb{Q}$ elliptic curve of conductor $N = Mp$, $p \nmid M$.
- E is modular $\rightsquigarrow f = \sum a_n(f)q^n \in S_2(\Gamma_0(N))$ normalized eigenform.

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- $\mathcal{X} := \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) \simeq \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$, p-adic weight space.
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Hida theory: $f \rightsquigarrow U \subset \mathcal{X}$ neighborhood of 2 and formal q-expansion

$$f_\infty = \sum_{n \geq 1} a_n q^n, \quad a_1 = 1, \quad a_n \in \mathcal{A}(U),$$

such that

a) for all $k \in U \cap \mathbb{Z}^{\geq 2}$, $f_k = \sum a_n(k) q^n \in S_k(\Gamma_0(N))$ is a normalized *ordinary* eigenform;

b) $f_2 = f$.

p-adic L-functions: M–Sw-D and M–K

Attached to each f_k and the choice of a complex period Ω_{f_k}, we have seen the Mazur–Swinnerton-Dyer p-adic L-function

$$L_p(f_k, s).$$

When varying $k \in U \cap \mathbb{Z}^{\geq 2}$, these can be packaged into the two-variable Mazur–Kitagawa p-adic L-function

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This function is analytic in a neighborhood of $(2, 1)$, and for each $k \in U \cap \mathbb{Z}^{\geq 2}$ there is $\lambda(k) \in \mathbb{C}_p$ such that

$$L_p(f_\infty; k, s) = \lambda(k) L_p(f_k, s)$$

and $\lambda(2) = 1$. In particular, $L_p(f_\infty; 2, s) = L_p(f, s) = L_p(E, s)$.

Exceptional zeroes

There is a functional equation

$$L_p(f_\infty; k, s) \leftrightarrow L_p(f_\infty; k, k-s)$$

whose sign does **not** depend on k, $\mathrm{sign}(f_\infty) = \pm 1$.

Exceptional zero phenomenon: if p is of **split** multiplicative reduction,

$$\mathrm{sign}(f_\infty) = -\mathrm{sign}(E/\mathbb{Q}).$$

Exceptional zero conjecture: $L_p'(f, 1) = \frac{\log(q)}{\mathrm{ord}_p(q)} \frac{L(f,1)}{\Omega_f}$.

Sign dichotomy

- sign($E/\mathbb{Q}$) $= +1$ (i.e sign(f_∞) $= -1$): $L_p(f_\infty; k, s)_{|s=k/2} = 0$, hence

$$\frac{\partial}{\partial s} L_p(f_\infty; 2, 1) = -2 \frac{\partial}{\partial k} L_p(f_\infty; 2, 1).$$

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Exploiting this and a factorization of $L_p(f_\infty; k, s)_{|s=1}$, Greenberg and Stevens show that

$$L_p'(f, 1) = -2a_p'(2) \frac{L(f, 1)}{\Omega_f},$$

and by studying the Λ-adic representation attached to f_∞, that

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$$\frac{\partial}{\partial s} L_p(f_\infty; 2, 1) = \frac{\partial}{\partial k} L_p(f_\infty; 2, 1) = 0,$$

hence $\mathrm{ord}_{(k,s)=(2,1)} L_p(f_\infty; k, s) \geq 2$.

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  What about the second derivative?

A second derivative result

Let $\log_E : E(\mathbb{Q}_p) \to \mathbb{G}_a(\mathbb{Q}_p)$ be the formal group logarithm,

$$\log_E(P) := \log_q(\Phi_{\mathrm{Tate}}^{-1}(P)),$$

where $\log_q : \mathbb{C}_p^\times \to \mathbb{C}_p$ (with $\log_q(q) = 0$), $\Phi_{\mathrm{Tate}} : \mathbb{C}_p^\times \to E(\mathbb{C}_p)$.

Theorem (Bertolini–Darmon)

Suppose E has split *multiplicative reduction at p and has at least two primes of multiplicative reduction.*

1) *There is a global point $\mathbf{P} \in E(\mathbb{Q}) \otimes \mathbb{Q}$ and a scalar $C \in \mathbb{Q}^\times$ such that*

$$\frac{d^2}{dk^2} L_p(f_\infty; k, k/2)_{|k=2} = C \cdot \log_E(\mathbf{P})^2.$$

2) *The point $\mathbf{P}$ is of infinite order if and only if $L'(E,1) \neq 0$.*

Modular symbols

$\mathfrak{P}_k = \mathfrak{P}_k(\mathbb{C}_p) =$ degree $k-2$ homogeneous polynomials in X, Y.
$V_k = V_k(\mathbb{C}_p) = \mathrm{Hom}_{\mathbb{C}_p}(\mathfrak{P}_k, \mathbb{C}_p)$ its $\mathbb{C}_p$-dual.

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A normalized cusp form $g \in S_k(\Gamma_0(N))$ gives rise to a basic modular symbol $\tilde{I}_g \in MS(V_k(\mathbb{C}))$,

$$\tilde{I}_g : \mathbb{P}_1(\mathbb{Q}_p) \times \mathbb{P}_1(\mathbb{Q}_p) \to V_k(\mathbb{C}),\ \tilde{I}_g\{r \to s\}(P) = 2\pi i \int_r^s g(z)P(z,1)dz.$$

Shimura: there exist complex periods Ω_g^+, Ω_g^- such that

$$I_g^+ := \frac{\tilde{I}_g^+}{\Omega_g^+}, \quad I_g^- := \frac{\tilde{I}_g^-}{\Omega_g^-} \in MS(V_k(K_g)).$$

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We fix $I_g := I_g^{w_\infty}$, $\Omega_g := \Omega_g^{w_\infty}$ for some choice $w_\infty \in \{\pm 1\}$.

Modular symbols and special values

Let $j \in \mathbb{Z}$ be such that $1 \leq j \leq k-1$. For $a, m \in \mathbb{Z}$, put

$$I_g[j,a,m] := I_g\{\infty \to a/m\}((x - \frac{a}{m}y)^{j-1}y^{k-j-1}).$$

The values $I_g[j,a,m]$ depend only on a modulo m, and they are related to special values of L-series: if χ is a primitive Dirichlet character modulo m and $\chi(-1) = (-1)^{j-1}w_\infty$, then

$$L^*(g,\chi,j) := \frac{(j-1)!\tau(\chi)}{(-2\pi i)^{j-1}\Omega_g} L(g,\chi,j) \in K_g$$

and

$$L^*(g,\chi,j) = \sum_{a=1}^{m} \chi(a) I_g[j,a,m].$$

In particular, when $(-1)^{j-1} = w_\infty$ we have $L^*(g,j) = I_g[j,1,1]$.

Hida theory

$\Lambda := \mathbb{Z}_p[[1 + p\mathbb{Z}_p]] \subset \tilde{\Lambda} := \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ usual Iwasawa algebras.
View $\tilde{\Lambda}$ as space of functions on weight space

$$\mathcal{X} := \mathrm{Hom}_{\mathrm{cont}}(\tilde{\Lambda}, \mathbb{Z}_p) = \mathrm{Hom}_{\mathrm{grp}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) \simeq \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p.$$

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The algebra homomorphism $\eta_f : \mathbb{T}_\infty^{\mathrm{ord}} \longrightarrow \mathbb{Z}_p$

$$T_n \longmapsto a_n(f), U_p \longmapsto a_p(f)$$

gives rise to the Hida family passing through f:

$$f_\infty := \sum_{n\geq 1} a_n q^n, \quad a_n := \eta_{f_\infty}(T_n) \in \Lambda^\dagger.$$

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For $k \in U \cap \mathbb{Z}^{\geq 2}$, $f_k := \sum a_n(k) q^n \in S_k(\Gamma_0(N))$ and $f_2 = f$. But for $k > 2$, f_k is not new: it is the ordinary p-stabilization of $f_k^\sharp \in S_k(\Gamma_0(N/p))$.

Measure-valued modular symbols

$L_* := \mathbb{Z}_p^2 \subset \mathbb{Q}_p^2$ standard $\mathbb{Z}_p$-lattice, $L'_* :=$ subset of primitive vectors.
The space of measures on L'_*,

$$\mathbb{D}_* := \mathrm{Hom}_{\mathrm{cont}}(L'_*, \mathbb{C}_p)^\vee = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Hom}_{\mathrm{cont}}(L'_*, \mathbb{C}_p), \mathbb{C}_p),$$

has a left action of $\mathrm{GL}_2(\mathbb{Z}_p)$, and a $\tilde{\Lambda}$-module structure induced by the action of $\mathbb{Z}_p^\times$ on L'_*.
For every $k \in \mathbb{Z}^{\geq 2}$ there is a $\Gamma_0(p\mathbb{Z}_p)$-equivariant homomorphism

$$\rho_k : \mathbb{D}_* \longrightarrow V_k$$

defined by the rule $\rho_k(\mu)(P) := \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} P(x, y) d\mu(x, y)$, inducing

$$\rho_k : MS_{\Gamma_0(N)}(\mathbb{D}_*) \longrightarrow MS_{\Gamma_0(N)}(V_k) \quad \text{(specialization map)}.$$

Measure-valued modular symbols

$MS_{\Gamma_0(N)}(\mathbb{D}_*)$ has a Hecke action (incl. U_p) compatible with maps ρ_k, and

$$MS^{\mathrm{ord}}_{\Gamma_0(N)}(\mathbb{D}_*) := e_{\mathrm{ord}} MS_{\Gamma_0(N)}(\mathbb{D}_*) \subset MS_{\Gamma_0(N)}(\mathbb{D}_*),$$
$$MS^{\mathrm{ord}}_{\Gamma_0(N)}(\mathbb{D}_*)^\dagger := MS^{\mathrm{ord}}_{\Gamma_0(N)}(\mathbb{D}_*) \otimes_\Lambda \Lambda^\dagger \subset MS_{\Gamma_0(N)}(\mathbb{D}^\dagger_*)$$

are free modules of finite rank over Λ, resp. $\Lambda^\dagger$.

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Theorem (Greenberg–Stevens)

There is a neighborhood U of $2 \in \mathcal{X}$ and a measure valued modular symbol $\mu_* \in MS^{ord}_{\Gamma_0(N)}(\mathbb{D}_*)^\dagger$, which is regular on U, such that

i) $\rho_2(\mu_*) = I_f$,
ii) *for all $k \in U \cap \mathbb{Z}^{>2}$, there exists a scalar $\lambda(k) \in \mathbb{C}_p$ such that* $\rho_k(\mu_*) = \lambda(k) I_{f_k}$.

Moreover, one can choose U so that $\lambda(k) \neq 0$ for all $k \in U \cap \mathbb{Z}^{\geq 2}$.

p-adic L-functions

The $\mathbb{D}_*^\dagger$-valued modular symbol μ_* can be used to attach a two-variable p-adic L-function of $(k, s) \in U \times \mathcal{X}$ to f_∞, namely the Mazur–Kitagawa p-adic L-function

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$$L_p(f_\infty; k, s).$$

Interpolation property: if $1 \leq j \leq k-1$ and $(-1)^j = w_\infty$, then

$$L_p(f_\infty; k, j) = \lambda(k)(1 - a_p(k)^{-1} p^{j-1}) L^*(f_k, j).$$

In terms of $f_k^\sharp$, for $j = k/2$ one finds

$$L_p(f_\infty; k, k/2) = \lambda(k)(1 - a_p(k)^{-1} p^{k/2-1})^2 L^*(f_k^\sharp, k/2).$$

Setting and motivation

Some preliminaries

Modular forms on quaternion algebras

Auxiliary p-adic L-functions

Heegner points

Proof of the main result

Automorphic forms on definite quaternion algebras

Set $N = pN^+N^-$, with $(N^+, N^-) = 1$, N^- = square-free product of an odd number of primes, and let

B = definite quaternion algebra of discriminant $N^-\infty$.

$\iota_p : B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p \to M_2(\mathbb{Q}_p)$ isomorphism.

$\Sigma = \prod_\ell \Sigma_\ell$ compact open subset (level structure) of $\hat{B}^\times$.

A = a $\mathbb{Q}_p$-vector space or $\mathbb{Z}_p$-module with left linear $\mathrm{GL}_2(\mathbb{Q}_p)$-action.

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Definition

$\phi : \hat{B}^\times \to A$ *is an A-valued automorphic form on $B^\times$ of level Σ if*

$$\phi(gb\sigma) = \iota_p(\sigma_p)^{-1}\phi(b) \quad \textit{for all } g \in B^\times, b \in \hat{B}^\times, \sigma \in \Sigma.$$

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Any A-valued automorphic form $\phi \in S(\Sigma; A)$ is determined by its values in a set of representatives for the (finite!) double coset

$$X_\Sigma := B^\times \backslash \hat{B}^\times / \Sigma.$$

Main examples

1) For $A = \mathbb{Z}_p$ or $\mathbb{Q}_p$ with trivial action, $S_2(\Sigma) := S(\Sigma; A)$ is the space of **weight two forms** of level Σ.
2) When $A = V_k(\mathbb{Q}_p)$, dual of $\mathfrak{P}_k(\mathbb{Q}_p)$, $S_k(\Sigma) := S(\Sigma; V_k(\mathbb{Q}_p))$ is the space of **weight k forms** of level Σ.
3) If $\Sigma_p = \mathrm{GL}_2(\mathbb{Q}_p)$ and $A = \mathbb{D}_*$ or $\mathbb{D}_*^\dagger$, one gets spaces of ***p*-adic families of modular forms**

$$S_\infty(\Sigma) := S(\Sigma; \mathbb{D}_*), \quad S_\infty^\dagger(\Sigma) := S(\Sigma; \mathbb{D}_*^\dagger).$$

These are modules over Λ and $\Lambda^\dagger$, with compatible Hecke action. When replacing Σ_p by $\Sigma'_p = \Gamma_0(p\mathbb{Z}_p)$, $\rho_k : \mathbb{D}_* \to V_k$ induces a $\Gamma_0(p\mathbb{Z}_p)$-equivariant weight k specialization homomorphism

$$\rho_k : S_\infty(\Sigma) \longrightarrow S_k(\Sigma').$$

Level structures

Let $\underline{R} \subset B$ be an Eichler order of level N^+ with $\iota_p(\underline{R} \otimes \mathbb{Z}_p) = M_2(\mathbb{Z}_p)$. Then we consider the level structure

$$\Sigma = \Sigma_0(N^+, N^-) = \prod_\ell (\underline{R} \otimes \mathbb{Z}_\ell)^\times \subset \hat{B}^\times$$

and the (finite) double coset

$$X_0(N^+, N^-) = B^\times \backslash \hat{B}^\times / \Sigma_0(N^+, N^-).$$

Notice that $\Sigma_p = \mathrm{GL}_2(\mathbb{Z}_p)$. Replacing Σ_p by $\Sigma'_p = \iota_p^{-1}(\Gamma_0(p\mathbb{Z}_p))$, strong approximation gives that for $\Sigma' = \Sigma_0(pN^+, N^-)$

$$X_0(pN^+, N^-) = R^\times \backslash B_p^\times / \iota_p^{-1}(\Gamma_0(p\mathbb{Z}_p)) = \tilde{\Gamma} \backslash \mathrm{GL}_2(\mathbb{Q}_p) / \Gamma_0(p\mathbb{Z}_p),$$

where R is an Eichler $\mathbb{Z}[1/p]$-order with $\underline{R}[1/p] = R$ and $\tilde{\Gamma} := \iota_p(R^\times)$.

Jacquet–Langlands

Recall our factorization $N = pM = pN^+N^-$.

If $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup, write $S_k(\Gamma) := S_k(\Gamma; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and similarly

$$S_k(N^+, N^-) := S_k(\Sigma_0(N^+, N^-); V_k(\mathbb{Q}_p)),$$
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Theorem

There are Hecke-equivariant isomorphisms

$$S_k(N^+, N^-) \to S_k^{new-N^-}(\Gamma_0(M)), \; S_k(pN^+, N^-) \to S_k^{new-N^-}(\Gamma_0(N)).$$

Crucial: $E/\mathbb{Q}$ has at least one prime $\neq p$ of multiplicative reduction!

Jacquet–Langlands in families

$$f \rightsquigarrow f_\infty \rightsquigarrow \{f_k^\sharp \in S_k^{\mathrm{new}-N^-}(\Gamma_0(M)) : k \in U \cap \mathbb{Z}^{>2}\}$$
$$(\mathrm{JL}) \rightsquigarrow \{\phi_k^\sharp \in S_k(N^+, N^-) : k \in U \cap \mathbb{Z}^{>2}\},$$
$$(p\text{-stab}) \rightsquigarrow \{\phi_k \in S_k(pN^+, N^-) : k \in U \cap \mathbb{Z}^{>2}\},\ \langle \phi_k, \phi_k \rangle = 1.$$

Also, $f = f_2 \rightsquigarrow \phi_2 \in S_2(pN^+, N^-)$, **new at** p. Require ϕ_2 to be $\mathbb{Z}$-valued.

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Also, $f = f_2 \rightsquigarrow \phi_2 \in S_2(pN^+, N^-)$, **new at** p. Require ϕ_2 to be $\mathbb{Z}$-valued.

Theorem

There is a p-adic family $\phi_\infty \in S_\infty^\dagger(N^+, N^-)$ *such that*

- $\rho_2(\phi_\infty) = \phi_2$;
- *if* U *is a neighborhood of regularity for the measures* $\phi_\infty(g) \in \mathbb{D}_*^\dagger$ *and* $k \in U \cap \mathbb{Z}^{\geq 2}$, *there is a scalar* $\lambda_B(k) \in \mathbb{C}_p$ *such that*

$$\rho_k(\phi_\infty) = \lambda_B(k)\phi_k.$$

Remarks

- The forms ϕ_k, $\phi_k^\sharp$ can be seen as functions on lattices in $\mathbb{Q}_p^2$ satisfying certain invariance and compatibility properties.

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- Further, one can associate to ϕ_∞ a collection of measures $\mu_L \in \mathbb{D}^\dagger$ indexed by the lattices in $\mathbb{Q}_p^2$ (in a similar way as for the measure-valued modular symbol μ_*).

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- Further, one can associate to ϕ_∞ a collection of measures $\mu_L \in \mathbb{D}^\dagger$ indexed by the lattices in $\mathbb{Q}_p^2$ (in a similar way as for the measure-valued modular symbol μ_*).
- In weight 2, the situation is particularly nice and explicit: $\phi = \phi_2$ might be seen as a ($\mathbb{Z}$-valued) function on the Bruhat–Tits tree, and defines a *boundary measure* μ_ϕ on $\mathbb{P}_1(\mathbb{Q}_p) = \partial\mathcal{H}_p$.
  Teitelbaum associates to μ_ϕ a rigid analytic modular form

$$f_\phi(z) = \int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{d\mu_\phi(t)}{(t-z)}$$

  on $\mathcal{H}_p$, invariant under $\Gamma := \{\gamma \in \tilde{\Gamma} : \det(\gamma) = 1\} \subset \mathrm{SL}_2(\mathbb{Q}_p)$.

Aim:

to introduce an **auxiliary p-adic L-function**

$$L_p(f_\infty/K;k)$$

associated with f_∞ and a choice of (admissible) imaginary quadratic field K, by means of **optimal embeddings** of $\mathcal{O}_K$ into a suitable definite quaternion algebra, and relate it to the **Mazur–Kitagawa p-adic L-function**

$$L_p(f_\infty;k,s).$$

Optimal embeddings

Set $N = pN^+N^-$, with $(N^+, N^-) = 1$, N^- = square-free product of an odd number of primes, and let

B = definite quaternion algebra of discriminant $N^-\infty$.

R = Eichler $\mathbb{Z}[1/p]$-order of level N^+.

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Let K be an *admissible* imaginary quadratic field, i.e.

i) all primes dividing N^+, resp. N^-, are split, resp. inert, in K;

ii) the prime p is unramified in K.

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i) all primes dividing N^+, resp. N^-, are split, resp. inert, in K;

ii) the prime p is unramified in K.

Definition

A pair $(\Psi, b) \in \mathrm{Hom}(K, B) \times (\hat{B}^\times / \hat{\underline{R}}^\times)$ *is an* optimal embedding *of* K *into* B *of level* N^+ *if*

$$\Psi(K) \cap (b\hat{\underline{R}}b^{-1}) = \Psi(\mathcal{O}_K).$$

Optimal embeddings: $B^\times$- and G_D-actions

$B^\times$ acts on the left on $\mathrm{Hom}(K, B) \times (\hat{B}^\times / \underline{\hat{R}}^\times)$ by

$$g \cdot (\Psi, b) := (g\Psi g^{-1}, gb)$$

preserving the set of optimal embeddings of level N^+.

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$$\mathrm{Emb}(\mathcal{O}_K, N^+, N^-) = \text{set of orbits} = B^{\times}\backslash\{\text{Optimal embeddings}\}.$$

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$$\mathrm{Emb}(\mathcal{O}_K, N^+, N^-) = \text{set of orbits} = B^\times \backslash \{\text{Optimal embeddings}\}.$$

The class group $G_D := \hat{K}^\times \backslash K^\times \hat{\mathcal{O}}_K^\times$ acts on $\mathrm{Emb}(\mathcal{O}_K, N^+, N^-)$ by

$$[\Psi^\sigma] := \sigma \cdot [\Psi, b] := [\Psi, \hat{\Psi}(\sigma) b],$$

where $\hat{\Psi} : \hat{K} \to \hat{B}$ is induced by $\Psi : K \to B$ (tensoring with $\hat{\mathbb{Z}}$).

Optimal embeddings: quadratic forms

An optimal embedding $\Psi = (\Psi, b)$ determines a quadratic form of discriminant $4D$ and $\mathbb{Q}_p$-coefficients,

$$Q_\Psi(x,y) = cx^2 + (d-a)xy - by^2, \quad \text{where } \iota_p\Psi(\sqrt{D}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Up to sign, Q_Ψ is determined by its discriminant and the property

$$Q_\Psi(\tau_\Psi, 1) = Q_\Psi(\tau'_\Psi, 1) = 0,$$

where τ_Ψ, $\tau'_\Psi \in \mathbb{P}_1(\mathbb{C}_p)$ are the fixed points of $\iota_p(\Psi(K_p^\times))$.

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Associated to $\phi_k^\sharp \in S_k(N^+, N^-)$ and $[\Psi] = [\Psi, b]$, define

$$\phi_k^\sharp[\Psi] := \det(b_p)^{1-k/2}\phi_k^\sharp(b)(Q_\Psi^{\frac{k-2}{2}}|b_p) \in \mathbb{C}_p.$$

If p splits in K and $[\Psi, b] \in \mathrm{Emb}(\mathcal{O}_K, pN^+, N^-)$, define $\phi_2[\Psi] := \phi_2(b)$.

Optimal embeddings and special values ($k > 2$)

Recall the modular forms $f_k^\sharp$. The algebraic part of $L(f_k^\sharp/K, k/2)$ is

$$L^*(f_k^\sharp/K, k/2) := \frac{(k/2-1)!^2 D^{\frac{k-1}{2}}}{(2\pi)^{k-2}\langle f_k^\sharp, f_k^\sharp\rangle} \cdot L(f_k^\sharp/K, k/2).$$

For $k = 2$, $L^*(f/K, 1) = \langle f, f\rangle^{-1}\sqrt{D} \cdot L(f/K, 1)$.

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Proposition (Hatcher, Hui Xue)

For all $k \in U \cap \mathbb{Z}^{>2}$,

$$L^*(f_k^\sharp/K, k/2) = \left(\sum_{j=1}^{h} \phi_k^\sharp[\Psi^{\sigma_j}]\right)^2.$$

Optimal embeddings and special values ($k = 2$)

When $k = 2$, we consider $f = f_2$ and ϕ_2 (of level pN^+N^-).
Observe $\mathrm{Emb}(\mathcal{O}_K, pN^+, N^-)$ is non-empty if and only if p *splits* in K.

Proposition

$$L^*(f/K, 1) = \begin{cases} 0 & \text{if } p \text{ is inert in } K, \\ \langle \phi_2, \phi_2 \rangle^{-1} \left(\sum_{j=1}^{h} \phi_2[\Psi^{\sigma_j}] \right)^2 & \text{if } p \text{ splits in } K. \end{cases}$$

(Two-variable) p-adic L-functions

Fix $2 \in U \subset \mathcal{X}$ as usual.

Definition

The p-adic partial L-function attached to f_∞/K and $[\Psi]$ *is*

$$\mathcal{L}_p(f_\infty/K, \Psi; k) := \int_{\tilde{L}_\Psi} \langle Q_\Psi(x,y)\rangle^{\frac{k-2}{2}} d\mu_{L_\Psi}(x,y).$$

The p-adic L-function attached to f_∞/K *is*

$$L_p(f_\infty/K; k) := (\mathcal{L}_p(f_\infty/K; k))^2,$$

where

$$\mathcal{L}_p(f_\infty/K; k) := \sum_{j=1}^{h} \mathcal{L}_p(f_\infty/K, \Psi^{\sigma_j}; k).$$

Theorem (Interpolation property)

- *If p is inert in K, then* $L_p(f_\infty/K, 2) = 0$ *and for* $k \in U \cap \mathbb{Z}^{>2}$

$$L_p(f_\infty/K; k) = \lambda_B(k)^2 a_p(k)^2 \left(1 - \frac{p^{k-2}}{a_p(k)^2}\right)^2 L^*(f_k^\sharp/K, k/2).$$

- *If p splits in K, then*

$$L_p(f_\infty/K; 2) = (1 - a_p^{-1})^2 \langle \phi_2, \phi_2 \rangle L^*(f/K, 1),$$

and for all $k \in U \cap \mathbb{Z}^{>2}$,

$$L_p(f_\infty/K; k) = \lambda_B(k)^2 a_p(k)^2 \left(1 - \frac{p^{\frac{k-2}{2}}}{a_p(k)}\right)^4 L^*(f_k^\sharp/K, k/2)$$

$$= \lambda_B(k)^2 a_p(k)^2 \left(1 - \frac{p^{\frac{k-2}{2}}}{a_p(k)}\right)^2 L^*(f_k/K, k/2).$$

Remark

Assume $h = 1$ for simplicity. By factoring the quadratic form Q_Ψ as

$$Q_\Psi(x, y) = A(x - \tau_\Psi y)(x - \tau'_\Psi y),$$

one could define a two-variable p-adic L-function on $(k, s) \in U \times \mathbb{Z}_p$ by

$$\mathcal{L}_p(f_\infty/K; k, s) := A^{\frac{k-2}{2}} \int_{\tilde{L}_\Psi} \langle x - \tau_\Psi y\rangle^{s-1} \langle x - \tau'_\Psi y\rangle^{k-s-1} d\mu_{L_\Psi}(x, y),$$
$$L_p(f_\infty/K; k, s) := \mathcal{L}_p(f_\infty/K; k, s)\mathcal{L}_p(f_\infty/K; k, k - s).$$

When restricted to the "critical line" $s = k/2$ one recovers $L_p(f_\infty/K; k)$. And when restricted to the "weight 2 line" $k = 2$, $L_p(f_\infty/K; 2, s)$ is the **anticyclotomic p-adic L-function** interpolating special values of the L-series attached to E/K, twisted by ring class characters of p-power conductor (Bertolini–Darmon).

A factorization of p-adic L-functions

Consider the function

$$\eta(k) := \begin{cases} \langle \phi_2, \phi_2 \rangle & \text{if } k = 2, \\ a_p(k)^2 D^{k/2-1} \frac{\lambda_B(k)^2}{\lambda^+(k)\lambda^-(k)} & \text{for } k \in U \cap \mathbb{Z}^{>2}. \end{cases}$$

Proposition

For every $k \in U \cap \mathbb{Z}^{\geq 2}$,

$$L_p(f_\infty/K; k) = \eta(k) L_p(f_\infty; k, k/2) L_p(f_\infty, \epsilon_K; k, k/2).$$

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$$L_p(f_\infty/K; k) = \eta(k) L_p(f_\infty; k, k/2) L_p(f_\infty, \epsilon_K; k, k/2).$$

The proof follows from the complex factorization of special values

$$L(f_k^\sharp/K, k/2) = L(f_k^\sharp, k/2) L(f_k^\sharp, \epsilon_K, k/2)$$

and the interpolation properties.

Derivatives

Assume p is **inert** in K. Then for any $[\Psi]$ we have $\mathcal{L}_p(f_\infty/K,\Psi;2)=0$. Let τ_Ψ,τ'_Ψ be the fixed points of Ψ and set

$$J_\Psi := \int^{\tau_\Psi} \omega_\phi, \quad \bar{J}_\Psi := \int^{\tau'_\Psi} \omega_\phi,$$

where for $\tau \in \mathcal{H}_p(\mathbb{Q}_{p^2})$, $L_\tau := \{(x,y)\in\mathbb{Q}_p^2 : x-\tau_\Psi y \in \mathbb{Z}_{p^2}\}$ and

$$\int^{\tau}\omega_\phi := \int_{L'_\tau} \log(x-\tau_\Psi y) d\mu_{L_\tau} := \frac{d}{dk}\left(\int_{L'_\tau}\langle x-\tau y\rangle^{k-2} d\mu_{L_\tau}\right)_{k=2}$$

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Theorem

For all $[\Psi]\in\mathrm{Emb}(\mathcal{O}_K,N^+,N^-)$,

$$\frac{d}{dk}\mathcal{L}_p(f_\infty/K,\Psi;k)_{k=2} = \frac{1}{2}(J_\Psi+\bar{J}_\Psi).$$

Aim:

to intepret via Cherednik-Drinfeld the previous **optimal embeddings** as special points on an indefinite Shimura curve, and deduce a relationship between the second derivative of $L_p(f_\infty/K;k)$ at $k=2$ and a **Heegner point** on E.

Shimura curves

Recall the Eichler $\mathbb{Z}[1/p]$-order R of level N^+ in B, $\tilde{\Gamma} = \iota_p(R^\times)$ and

$$\Gamma = \{\gamma \in \tilde{\Gamma} : \det(\gamma) = 1\} \subset \mathrm{SL}_2(\mathbb{Q}_p).$$

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The weight two form ϕ defines the rigid analytic modular form

$$f_\phi(z) = \int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{d\mu_\phi(t)}{(t-z)} \quad \text{(Teitelbaum)},$$

which gives a Γ-invariant differential $\omega_\phi = f_\phi(z)dz$ on the p-adic upper half plane $\mathcal{H}_p = \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$, or equivalently a differential on

$$X_\Gamma = \mathcal{H}_p/\Gamma.$$

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Mumford $\Rightarrow$ $X_\Gamma = X_{\mathbb{Q}_p}(\mathbb{C}_p)$ for some projective curve over $\mathbb{Q}_p$.

Cherednik–Drinfeld

Let $\mathcal{B}$ be the *indefinite* quaternion algebra of discriminant pN^- obtained from B by interchanging local invariants at p and ∞. Fix an isomorphism

$$\iota_\infty : \mathcal{B} \otimes \mathbb{R} \longrightarrow M_2(\mathbb{R}),$$

an Eichler order $\mathcal{S}$ of level N^+, and let $\Gamma_\infty := \iota_\infty(\mathcal{S}_1) \subset \mathrm{SL}_2(\mathbb{R})$. By Shimura, the compact Riemann surface

$$X_{\Gamma_\infty} := \mathcal{H}/\Gamma_\infty$$

is the complex analytic space attached to a projective curve $X/\mathbb{Q}$: the coarse moduli space for QM-abelian surfaces with level N^+ structure.

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Theorem (Cherednik–Drinfeld)

X_Γ is the rigid analytic space attached to $X(\mathbb{C}_p)$. More precisely, X and $X_{\mathbb{Q}_p}$ are isomorphic over $X_{\mathbb{Q}_{p^2}}$.

Modular parametrizations

Let $\mathrm{Jac}_X = \mathrm{Div}^0(X)/P(X)$ be the Jacobian of X.

$$\left.\begin{array}{r}\text{Jacquet–Langlands}\\ \text{Eichler–Shimura}\end{array}\right\} \Rightarrow \mathrm{Jac}_X \text{ has simple factor } A_\phi, L(A_\phi, s) = L(E, s).$$

By the Isogeny Theorem, $A_\phi \sim_{\mathbb{Q}} E$, hence $\exists\ \varphi_E : \mathrm{Jac}_X \twoheadrightarrow E$.

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By the Isogeny Theorem, $A_\phi \sim_{\mathbb{Q}} E$, hence $\exists\ \varphi_E : \mathrm{Jac}_X \twoheadrightarrow E$.
Let $\mathrm{Div}^0(X_\Gamma)$ = degree zero divisors supported on $X_\Gamma = \mathcal{H}_p/\Gamma$, and

$$\mathrm{Jac}_\Gamma := \mathrm{Div}^0(X_\Gamma)/P(X_\Gamma).$$

Identifying Jac_Γ with $\mathrm{Jac}_X(\mathbb{C}_p)$ via Cherednik–Drinfeld:

Theorem

Over $\mathbb{Q}_{p^2}$, $\varphi_E : \mathrm{Jac}_X \to E$ can be described by $\varphi_E(d) = \Phi_{Tate}\left(\times\!\!\!\!\!\!\int_d \omega_\phi\right)$.

Heegner points

From now on we assume p **inert** in K.

$H := \mathrm{Hilb}(K)$, $G_D \simeq \mathrm{Gal}\,(H/K)$, $H \hookrightarrow \mathbb{Q}_{p^2}$.
$[\Psi] \in \mathrm{Emb}(\mathcal{O}_K, N^+, N^-)$.

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$\iota_p \Psi(K_p^\times)$ has a unique fixed point $\tau_\Psi \in \mathcal{H}_p(\mathbb{Q}_{p^2})$ such that

$$\iota_p \Psi(\alpha)\begin{pmatrix} \tau_\Psi \\ 1 \end{pmatrix} = \alpha\begin{pmatrix} \tau_\Psi \\ 1 \end{pmatrix}.$$

By passing to the quotient,

$$\tau_\Psi \mapsto \tilde{P}_\Psi \in X_\Gamma(\mathbb{Q}_{p^2}) = X(\mathbb{Q}_{p^2}).$$

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Theorem (Complex Multiplication)

The point $\tilde{P}_\Psi$ belongs to $X(H)$, and for every $\sigma \in G_D$, $\tilde{P}_{\Psi^\sigma} = \sigma(\tilde{P}_\Psi)$.

Heegner points

Choose an auxiliary prime $\ell \nmid N$, and set

$$\tilde{P}_K := (\ell + 1 - a_\ell)^{-1}(\ell + 1 - T_\ell) \sum_{\sigma \in G_D} \tilde{P}_{\Psi^\sigma} \in \mathrm{Div}^0(X)(K) \otimes \mathbb{Q}.$$

Via φ_E we obtain the so-called Heegner point attached to Ψ,

$$P_K := \varphi_E(\tilde{P}_K) \in E(K) \otimes \mathbb{Q}.$$

Similarly, let $J_K := \sum_{\sigma \in G_D} J_{\Psi^\sigma}$, and $\bar{J}_K$ be its $\mathbb{Q}_{p^2}$-conjugate.

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Similarly, let $J_K := \sum_{\sigma \in G_D} J_{\Psi^\sigma}$, and $\bar{J}_K$ be its $\mathbb{Q}_{p^2}$-conjugate.

Theorem

(1) [Zhang] *P_K is of infinite order if and only if $L'(E/K, 1) \neq 0$. And in that case, $\mathrm{rk}_{\mathbb{Z}} E(K) = 1$.*

(2) *$J_K = \log_E(P_K)$ and $\bar{J}_K = a_p \log_E(P_K)$.*

Heegner points and derivatives

Using the relation of the periods J_K and $\bar{J}_K$ with the derivative of $\mathcal{L}_p(f_\infty/K;k)$ at $k=2$ and part (2) of the previous theorem,

$$\frac{d}{dk}\mathcal{L}_p(f_\infty/K;k)_{|k=2} = \frac{1}{2}(1+a_p)\log_E(P_K) = \begin{cases} \log_E(P_K) & \text{if } a_p = 1, \\ 0 & \text{if } a_p = -1. \end{cases}$$

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Corollary

$$\frac{d^2}{dk^2}L_p(f_\infty/K;k)_{|k=2} = \begin{cases} 2\log_E(P_K)^2 & \textit{if } a_p = 1 \\ 0 & \textit{if } a_p = -1 \end{cases}$$

Recall that $L_p(f_\infty/K;k) := (\mathcal{L}_p(f_\infty/K;k))^2$ and p is inert in K.

Theorem (Bertolini–Darmon)

Suppose E has split *multiplicative reduction at p, has at least two primes of multiplicative reduction, and sign$(E, \mathbb{Q}) = -1$.*

1) *There is a global point $\mathbf{P} \in E(\mathbb{Q}) \otimes \mathbb{Q}$ and a scalar $C \in \mathbb{Q}^\times$ such that*

$$\frac{d^2}{dk^2} L_p(f_\infty; k, k/2)_{|k=2} = C \cdot \log_E(\mathbf{P})^2.$$

2) *The point $\mathbf{P}$ is of infinite order if and only if $L'(E, 1) \neq 0$.*

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Sketch of proof: the assumptions imply that

$$\mathrm{ord}_{k=2} L_p(f_\infty; k, k/2) \geq 2.$$

Factor $N = pN^+N^-$ as before and choose K admissible such that p is inert in K and

$$L(f, \epsilon_K, 1) \neq 0.$$

Now use the factorization of p-adic L-functions

$$L_p(f_\infty/K;k) = \eta(k)L_p(f_\infty;k,k/2)L_p(f_\infty,\epsilon_K;k,k/2), \quad k \in U \cap \mathbb{Z}^{\geq 2}.$$

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Taking second derivatives and evaluating at $k=2$,

$$2\log_E(P_K)^2 = \eta(2)\frac{d^2}{dk^2}L_p(f_\infty;k,k/2)_{|k=2}L_p(f,\epsilon_K;1),$$

hence

$$\frac{d^2}{dk^2}L_p(f_\infty;k,k/2)_{|k=2} = \left(\eta(2)L^*(f,\epsilon_K,1)\right)^{-1}\cdot\log_E(P_K)^2.$$

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By our choice of K, Kolyvagin's theorem implies $P_K \in E(\mathbb{Q}) \otimes \mathbb{Q}$, so the theorem follows with $\mathbf{P} = P_K$.

A second derivative result for p-adic L-functions (after Bertolini–Darmon)

Carlos de Vera Piquero (Essen)

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