Venkatesh's conjecture for modular forms of weight one

Victor Rotger, UPC, Barcelona

STNB 2023

Venkatesh's conjecture

- G, a semisimple algebraic group over $\mathbb{Q}$.
- $K = K_f \cdot K_\infty$, compact open in $G(\mathbb{A})$, maximal at ∞.
- $Y = Y(K)$, the resulting arithmetic manifold.
- $\delta = \mathrm{rank}\, G(\mathbb{R}) - \mathrm{rank}\, K_\infty$.
- q, such that $2q + \delta = \dim Y$.

Examples

- For $G = \mathrm{SL}_2/\mathbb{Q}$ we have $\delta = 0$ and $q = 1$.
- For $G = \mathrm{Res}_{E/\mathbb{Q}}(\mathrm{SL}_2/E)$ with $E = \mathbb{Q}(\sqrt{-D})$ we have

$$\delta = 1, \quad q = 1.$$

Hecke systems of eigenvalues

- Denote $\mathbb{T} \subset \mathrm{End}\, H^i(Y, \mathbb{Q})$, Hecke algebra of level K.
- Fix $\chi : \mathbb{T} \to \mathbb{Z}$, a tempered system of eigenvalues.

Hecke systems of eigenvalues

- Denote $\mathbb{T} \subset \mathrm{End}\, H^i(Y, \mathbb{Q})$, Hecke algebra of level K.
- Fix $\chi : \mathbb{T} \to \mathbb{Z}$, a tempered system of eigenvalues.
- $H^i(Y, \mathbb{Q})_\chi := \{f \in H^i(Y, \mathbb{Q}) : T(f) = \chi(T) \cdot f \text{ for all } T \in \mathbb{T}\}$.

Hecke systems of eigenvalues

- Denote $\mathbb{T} \subset \mathrm{End}\, H^i(Y,\mathbb{Q})$, Hecke algebra of level K.
- Fix $\chi : \mathbb{T} \to \mathbb{Z}$, a tempered system of eigenvalues.
- $H^i(Y,\mathbb{Q})_\chi := \{f \in H^i(Y,\mathbb{Q}) : T(f) = \chi(T)\cdot f \text{ for all } T \in \mathbb{T}\}$.
- **(Borel)** $H^i(Y,\mathbb{Q})_\chi = 0$ for $i \notin [q, q+\delta]$.
- For $0 \le i \le \delta$,

$$\dim H^{q+i}(Y)_\chi = \binom{\delta}{i} \dim H^q(Y)_\chi.$$

Derived Hecke algebra

Fix a good prime p. Venkatesh has introduced a derived Hecke algebra

$$\widetilde{\mathbb{T}} \subseteq \mathrm{End}(H^*(Y, \mathbb{Z}_p)).$$

- $\widetilde{\mathbb{T}}$ is a graded commutative algebra $\widetilde{\mathbb{T}} = \bigoplus_{j \in \mathbb{Z}} \widetilde{\mathbb{T}}_j$.

Derived Hecke algebra

Fix a good prime p. Venkatesh has introduced a derived Hecke algebra

$$\widetilde{\mathbb{T}} \subseteq \mathrm{End}(H^*(Y, \mathbb{Z}_p)).$$

- $\widetilde{\mathbb{T}}$ is a graded commutative algebra $\widetilde{\mathbb{T}} = \bigoplus_{j \in \mathbb{Z}} \widetilde{\mathbb{T}}_j$.
- $\widetilde{\mathbb{T}}_0 = \mathbb{T}$.
- $\widetilde{\mathbb{T}}_j$ moves H^q to H^{q+j}

Derived Hecke algebra

Fix a good prime p. Venkatesh has introduced a derived Hecke algebra

$$\widetilde{\mathbb{T}} \subseteq \mathrm{End}(H^*(Y, \mathbb{Z}_p)).$$

- $\widetilde{\mathbb{T}}$ is a graded commutative algebra $\widetilde{\mathbb{T}} = \bigoplus_{j \in \mathbb{Z}} \widetilde{\mathbb{T}}_j$.
- $\widetilde{\mathbb{T}}_0 = \mathbb{T}$.
- $\widetilde{\mathbb{T}}_j$ moves H^q to H^{q+j}
- $\widetilde{\mathbb{T}}$ does not necessarily preserve $\mathbb{Q}$-rational structures:

$$\widetilde{\mathbb{T}}_1 H^q(Y, \mathbb{Q}_p) \subset H^{q+1}(Y, \mathbb{Q}_p), \quad \widetilde{\mathbb{T}}_1 H^q(Y, \mathbb{Q}) \not\subset H^{q+1}(Y, \mathbb{Q}).$$

Derived Hecke algebra

Fix a good prime p. Venkatesh has introduced a derived Hecke algebra

$$\widetilde{\mathbb{T}} \subseteq \mathrm{End}(H^*(Y, \mathbb{Z}_p)).$$

- $\widetilde{\mathbb{T}}$ is a graded commutative algebra $\widetilde{\mathbb{T}} = \bigoplus_{j\in\mathbb{Z}} \widetilde{\mathbb{T}}_j$.
- $\widetilde{\mathbb{T}}_0 = \mathbb{T}$.
- $\widetilde{\mathbb{T}}_j$ moves H^q to H^{q+j}
- $\widetilde{\mathbb{T}}$ does not necessarily preserve $\mathbb{Q}$-rational structures:

$$\widetilde{\mathbb{T}}_1 H^q(Y, \mathbb{Q}_p) \subset H^{q+1}(Y, \mathbb{Q}_p), \quad \widetilde{\mathbb{T}}_1 H^q(Y, \mathbb{Q}) \not\subset H^{q+1}(Y, \mathbb{Q}).$$

A theorem of Venkatesh

Associated to χ there is a *p*-adic Galois representation ρ_χ.

A theorem of Venkatesh

Associated to χ there is a *p*-adic Galois representation ρ_χ.

$$S_p := H^1_f(\mathbb{Q}, \mathrm{ad}(\rho_\chi)^*(1))$$

Theorem (Venkatesh)

- S_p is free of rank δ over $\mathbb{Z}_p$.
- $\widetilde{\mathbb{T}}_\chi$ is generated as an exterior algebra in degree 1 by $S_p^\vee$.
- $H^*(Y, \mathbb{Z}_p)_{\bar{\chi}} \cong \widetilde{\mathbb{T}} \otimes_{\mathbb{T}} H^q(Y, \mathbb{Z}_p)_{\bar{\chi}}$.

- $M(\mathrm{ad}(\rho_\chi)^*)$: motive attached to $\mathrm{ad}(\rho_\chi)^*$.
- Motivic cohomology $H^i_{\mathrm{mot}}(M_{\mathbb{Z}}, \mathbb{Q}(j)) \subset H^i_{\mathrm{mot}}(M, \mathbb{Q}(j))$

- $M(\mathrm{ad}(\rho_\chi)^*)$: motive attached to $\mathrm{ad}(\rho_\chi)^*$.
- Motivic cohomology $H^i_{\mathrm{mot}}(M_{\mathbb{Z}}, \mathbb{Q}(j)) \subset H^i_{\mathrm{mot}}(M, \mathbb{Q}(j))$
- $S^\vee := H^1_{\mathrm{mot}}(M(\mathrm{ad}(\rho_\chi)^*)_{\mathbb{Z}}, \mathbb{Q}(1))^\vee$ acts on $H^*(Y, \mathbb{Q}_p)_{\bar{\chi}}$ via its p-adic étale realization.

Venkatesh's conjecture

- $M(\mathrm{ad}(\rho_\chi)^*)$: motive attached to $\mathrm{ad}(\rho_\chi)^*$.
- Motivic cohomology $H^i_{\mathrm{mot}}(M_{\mathbb{Z}}, \mathbb{Q}(j)) \subset H^i_{\mathrm{mot}}(M, \mathbb{Q}(j))$
- $S^\vee := H^1_{\mathrm{mot}}(M(\mathrm{ad}(\rho_\chi)^*)_{\mathbb{Z}}, \mathbb{Q}(1))^\vee$ acts on $H^*(Y, \mathbb{Q}_p)_{\bar{\chi}}$ via its p-adic étale realization.
- **Conjecture (Venkatesh)** $S^\vee$ preserves $H^*(Y, \mathbb{Q})_{\bar{\chi}}$.

Venkatesh's conjecture

- $M(\mathrm{ad}(\rho_\chi)^*)$: motive attached to $\mathrm{ad}(\rho_\chi)^*$.
- Motivic cohomology $H^i_{\mathrm{mot}}(M_{\mathbb{Z}}, \mathbb{Q}(j)) \subset H^i_{\mathrm{mot}}(M, \mathbb{Q}(j))$
- $S^\vee := H^1_{\mathrm{mot}}(M(\mathrm{ad}(\rho_\chi)^*)_{\mathbb{Z}}, \mathbb{Q}(1))^\vee$ acts on $H^*(Y, \mathbb{Q}_p)_{\bar{\chi}}$ via its p-adic étale realization.
- **Conjecture (Venkatesh)** $S^\vee$ preserves $H^*(Y, \mathbb{Q})_{\bar{\chi}}$.

Derived Hecke operators for classical modular forms

- $S_1(M) = H^0(X_1(M), \omega)$, space of cuspforms of weight 1
- Pick a pair of primes (N, p), $\quad N \nmid M, \quad p \,||\, N-1$.

Derived Hecke operators for classical modular forms

- $S_1(M) = H^0(X_1(M), \omega)$, space of cuspforms of weight 1
- Pick a pair of primes (N, p), $\quad N \nmid M, \quad p \,||\, N-1$.
- Fix $\log : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{Z}/p\mathbb{Z}$.

Derived Hecke operators for classical modular forms

- $S_1(M) = H^0(X_1(M), \omega)$, space of cuspforms of weight 1
- Pick a pair of primes (N, p), $\quad N \nmid M, \quad p \mid\mid N-1$.
- Fix $\log : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{Z}/p\mathbb{Z}$.
- Covering $X_1(N) \to X_0(N)$ with deck group $(\mathbb{Z}/N\mathbb{Z})^\times$ yields

$$\mathrm{Sh} \in H^1(X_0(N), \mathbb{Z}/p\mathbb{Z}).$$

Derived Hecke operators for classical modular forms

- $S_1(M) = H^0(X_1(M), \omega)$, space of cuspforms of weight 1
- Pick a pair of primes (N, p), $\quad N \nmid M, \quad p \,||\, N-1$.
- Fix $\log : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{Z}/p\mathbb{Z}$.
- Covering $X_1(N) \to X_0(N)$ with deck group $(\mathbb{Z}/N\mathbb{Z})^\times$ yields

$$\mathrm{Sh} \in H^1(X_0(N), \mathbb{Z}/p\mathbb{Z}).$$

- Forgetful maps $\pi_1, \pi_2 : X_{1,0}(M, N) \longrightarrow X_1(M)$

$$\begin{aligned} \tilde{T}_{N,\mathrm{Sh}} : H^0(\bar{X}_1(M), \omega) &\longrightarrow \quad H^1(\bar{X}_1(M), \omega) \pmod p \\ g &\mapsto \qquad \pi_{2,*}(\mathrm{Sh}_{MN} \cup \pi_1^*(g)) \end{aligned}$$

Harris-Venkatesh for modular forms of weight one

- $g \in S_1(M, \varepsilon), \quad g^* = g \otimes \varepsilon^{-1} \in S_1(M, \varepsilon^{-1})$

Harris-Venkatesh for modular forms of weight one

- $g \in S_1(M, \varepsilon), \quad g^* = g \otimes \varepsilon^{-1} \in S_1(M, \varepsilon^{-1})$
- $\langle \tilde{T}_{N,\mathrm{Sh}}(g), g^* \rangle$

Harris-Venkatesh for modular forms of weight one

- $g \in S_1(M, \varepsilon), \quad g^* = g \otimes \varepsilon^{-1} \in S_1(M, \varepsilon^{-1})$
- $\langle \tilde{T}_{N,\mathrm{Sh}}(g), g^* \rangle = \langle \mathrm{Sh}_{MN} \cup \pi_1^*(g), \pi_2^*(g^*) \rangle = \ldots$

Harris-Venkatesh for modular forms of weight one

- $g \in S_1(M, \varepsilon), \quad g^* = g \otimes \varepsilon^{-1} \in S_1(M, \varepsilon^{-1})$
- $\langle \tilde{T}_{N,\mathrm{Sh}}(g), g^* \rangle = \langle \mathrm{Sh}_{MN} \cup \pi_1^*(g), \pi_2^*(g^*) \rangle = ...$
- $... = \langle g(z) g^*(Nz), \mathrm{Sh}_{MN} \rangle$

Harris-Venkatesh for modular forms of weight one

- $g \in S_1(M, \varepsilon), \quad g^* = g \otimes \varepsilon^{-1} \in S_1(M, \varepsilon^{-1})$
- $\langle \tilde{T}_{N,\mathrm{Sh}}(g), g^* \rangle = \langle \mathrm{Sh}_{MN} \cup \pi_1^*(g), \pi_2^*(g^*) \rangle = \ldots$
- $\ldots = \langle g(z) g^*(Nz), \mathrm{Sh}_{MN} \rangle = \langle \mathrm{Tr}_N^{MN}(g(z) g^*(Nz)), \mathrm{Sh} \rangle$

Harris-Venkatesh for modular forms of weight one

- $H/\mathbb{Q}$: Galois extension cut out by $\mathrm{ad}(\rho_g)$
- $U_g := H^1_{\mathrm{mot}}(\mathrm{ad}(\rho_g)(1)) = \mathrm{Hom}_{G_{\mathbb{Q}}}(\mathrm{ad}(\rho_g), \mathcal{O}_H^\times) = \langle u_g \rangle$

- $H/\mathbb{Q}$: Galois extension cut out by $\mathrm{ad}(\rho_g)$
- $U_g := H^1_{\mathrm{mot}}(\mathrm{ad}(\rho_g)(1)) = \mathrm{Hom}_{G_{\mathbb{Q}}}(\mathrm{ad}(\rho_g), \mathcal{O}_H^\times) = \langle u_g \rangle$
- Reduction map $U_g \xrightarrow{\mathrm{red}_N} (\mathbb{Z}/N\mathbb{Z})^\times$

Harris-Venkatesh for modular forms of weight one

- $H/\mathbb{Q}$: Galois extension cut out by $\mathrm{ad}(\rho_g)$
- $U_g := H^1_{\mathrm{mot}}(\mathrm{ad}(\rho_g)(1)) = \mathrm{Hom}_{G_{\mathbb{Q}}}(\mathrm{ad}(\rho_g), \mathcal{O}_H^\times) = \langle u_g \rangle$
- Reduction map $U_g \xrightarrow{\mathrm{red}_N} (\mathbb{Z}/N\mathbb{Z})^\times$
- **Conjecture:**

$$\langle \mathrm{Tr}_N^{MN}(g(z)g^*(Nz)), \mathrm{Sh}\rangle \stackrel{?}{=} \log(\mathrm{red}_N(u_g)) \in \mathbb{Z}/p\mathbb{Z}$$

A theorem by Darmon-Harris-Rotger-Venkatesh

- Modular forms of weight 1 can either be *dihedral* or *exotic*.
- Dihedral: there is a quadratic field K and a character $\psi : G_K \longrightarrow L^\times$ such that

$$g = \theta(\psi), \quad \varrho_g = \mathrm{Ind}_{\mathbb{Q}}^{K}(\psi)$$

A theorem by Darmon-Harris-Rotger-Venkatesh

- Modular forms of weight 1 can either be *dihedral* or *exotic*.
- Dihedral: there is a quadratic field K and a character $\psi : G_K \longrightarrow L^\times$ such that

$$g = \theta(\psi), \quad \varrho_g = \mathrm{Ind}_{\mathbb{Q}}^{K}(\psi)$$

- **Theorem:** The Conjecture is true for g of dihedral type.

Proof when K is imaginary and N remains inert

- $\psi_g : G_K \longrightarrow L^\times, \quad g = \theta(\psi_g) \in S_1(M, \chi)$

Proof when K is imaginary and N remains inert

- $\psi_g : G_K \longrightarrow L^\times, \quad g = \theta(\psi_g) \in S_1(M, \chi)$
- $SS := \mathrm{Supersingular\, elliptic\, curves}\ /\bar{\mathbb{F}}_N$
- $\mathbb{X} := \mathrm{Hom}(SS, \mathbb{Z}/p\mathbb{Z})$

Proof when K is imaginary and N remains inert

- $\psi_g : G_K \longrightarrow L^\times, \quad g = \theta(\psi_g) \in S_1(M, \chi)$
- $SS := \text{Supersingular elliptic curves } /\bar{\mathbb{F}}_N$
- $\mathbb{X} := \mathrm{Hom}(SS, \mathbb{Z}/p\mathbb{Z})$
- $\Theta : \mathbb{X} \otimes_{\mathbb{T}} \mathbb{X} \longrightarrow M_2(N, \mathbb{Z}/p\mathbb{Z})$.
  Isomorphism locally at a Gorenstein maximal ideal of $\mathbb{T}$.

Proof when K is imaginary and N remains inert

- $\psi_g : G_K \longrightarrow L^\times, \quad g = \theta(\psi_g) \in S_1(M, \chi)$
- $SS := \mathrm{Supersingular\,elliptic\,curves}\ /\bar{\mathbb{F}}_N$
- $\mathbb{X} := \mathrm{Hom}(SS, \mathbb{Z}/p\mathbb{Z})$

- $\Theta : \mathbb{X} \otimes_{\mathbb{T}} \mathbb{X} \longrightarrow M_2(N, \mathbb{Z}/p\mathbb{Z})$.
  Isomorphism locally at a Gorenstein maximal ideal of $\mathbb{T}$.

- $\psi := \psi'_g/\psi_g : \mathrm{Pic}(\mathcal{O}) \longrightarrow \mathbb{Z}_L^\times \subset \mathbb{Z}_p^\times$

Proof when K is imaginary and N remains inert

- $\psi_g : G_K \longrightarrow L^\times, \quad g = \theta(\psi_g) \in S_1(M, \chi)$
- $SS := \text{Supersingular elliptic curves } /\bar{\mathbb{F}}_N$
- $\mathbb{X} := \mathrm{Hom}(SS, \mathbb{Z}/p\mathbb{Z})$
- $\Theta : \mathbb{X} \otimes_{\mathbb{T}} \mathbb{X} \longrightarrow M_2(N, \mathbb{Z}/p\mathbb{Z})$.
  Isomorphism locally at a Gorenstein maximal ideal of $\mathbb{T}$.
- $\psi := \psi'_g/\psi_g : \mathrm{Pic}(\mathcal{O}) \longrightarrow \mathbb{Z}_L^\times \subset \mathbb{Z}_p^\times$
- $\mathrm{Red_N} : \mathrm{Pic}(\mathcal{O}) \to S, \quad [\psi] \in \mathbb{X}$

Proof when K is imaginary and N remains inert

- $\psi_g : G_K \longrightarrow L^\times, \quad g = \theta(\psi_g) \in S_1(M, \chi)$
- $SS := \mathrm{Supersingular\,elliptic\,curves}\ /\bar{\mathbb{F}}_N$
- $\mathbb{X} := \mathrm{Hom}(SS, \mathbb{Z}/p\mathbb{Z})$
- $\Theta : \mathbb{X} \otimes_{\mathbb{T}} \mathbb{X} \longrightarrow M_2(N, \mathbb{Z}/p\mathbb{Z})$.
  Isomorphism locally at a Gorenstein maximal ideal of $\mathbb{T}$.
- $\psi := \psi'_g/\psi_g : \mathrm{Pic}(\mathcal{O}) \longrightarrow \mathbb{Z}_L^\times \subset \mathbb{Z}_p^\times$
- $\mathrm{Red_N} : \mathrm{Pic}(\mathcal{O}) \to S, \quad [\psi] \in \mathbb{X}$
- **Theorem A:** $\mathrm{Tr}_N^{MN}(g(z)g^*(Nz)) = \Theta([1] \otimes [\psi])$

Higher Eisenstein elements

- $\Sigma_0 \in \mathbb{X}, \quad E \in SS \mapsto \frac{1}{\#\mathrm{Aut}(E)}$, an Eisenstein element.
- Pick $\ell \nmid pMN, \quad u_\ell = \Delta(z)/\Delta(\ell z) \in \mathbb{F}_N(X)^\times$

Higher Eisenstein elements

- $\Sigma_0 \in \mathbb{X}, \quad E \in SS \mapsto \frac{1}{\#\mathrm{Aut}(E)}$, an Eisenstein element.
- Pick $\ell \nmid pMN, \quad u_\ell = \Delta(z)/\Delta(\ell z) \in \mathbb{F}_N(X)^\times$
- $\Sigma_1 : SS \hookrightarrow X(\mathbb{F}_{N^2}) \xrightarrow{u_\ell} \mathbb{F}_{N^2}^\times \xrightarrow{\log} \mathbb{Z}/p\mathbb{Z}$

Higher Eisenstein elements

- $\Sigma_0 \in \mathbb{X}, \quad E \in SS \mapsto \frac{1}{\#\mathrm{Aut}(E)}$, an Eisenstein element.
- Pick $\ell \nmid pMN, \quad u_\ell = \Delta(z)/\Delta(\ell z) \in \mathbb{F}_N(X)^\times$
- $\Sigma_1 : SS \hookrightarrow X(\mathbb{F}_{N^2}) \xrightarrow{u_\ell} \mathbb{F}_{N^2}^\times \xrightarrow{\log} \mathbb{Z}/p\mathbb{Z}$
- Extend $\mathrm{Sh} \in \mathrm{Hom}(S_2(N), \mathbb{Z}/p\mathbb{Z})$ to $M_2(N)^\vee$.

Higher Eisenstein elements

- $\Sigma_0 \in \mathbb{X}, \quad E \in SS \mapsto \frac{1}{\#\mathrm{Aut}(E)}$, an Eisenstein element.
- Pick $\ell \nmid pMN, \quad u_\ell = \Delta(z)/\Delta(\ell z) \in \mathbb{F}_N(X)^\times$
- $\Sigma_1 : SS \hookrightarrow X(\mathbb{F}_{N^2}) \xrightarrow{u_\ell} \mathbb{F}_{N^2}^\times \xrightarrow{\log} \mathbb{Z}/p\mathbb{Z}$
- Extend $\mathrm{Sh} \in \mathrm{Hom}(S_2(N), \mathbb{Z}/p\mathbb{Z})$ to $M_2(N)^\vee$.
- **Theorem B:** $\Theta^*(\mathrm{Sh}) = \Sigma_0 \otimes \Sigma_1 + \Sigma_1 \otimes \Sigma_0$

Conclusion of proof

- $G_N := \mathrm{Tr}\,(g(z)g^*(Nz)) \in S_2(N)$

Conclusion of proof

- $G_N := \mathrm{Tr}\,(g(z)g^*(Nz)) \in S_2(N)$
- $\langle G_N, \mathrm{Sh}\rangle = \langle [1]\otimes[\psi], \Sigma_0\otimes\Sigma_1 + \Sigma_1\otimes\Sigma_0\rangle$

Conclusion of proof

- $G_N := \mathrm{Tr}\,(g(z)g^*(Nz)) \in S_2(N)$
- $\langle G_N, \mathrm{Sh}\rangle = \langle [1] \otimes [\psi], \Sigma_0 \otimes \Sigma_1 + \Sigma_1 \otimes \Sigma_0\rangle$
- $\langle [1], \Sigma_0\rangle = h(\mathcal{O}), \quad \langle [\psi], \Sigma_0\rangle = 0$

Conclusion of proof

- $G_N := \mathrm{Tr}\,(g(z)g^*(Nz)) \in S_2(N)$
- $\langle G_N, \mathrm{Sh}\rangle = \langle [1] \otimes [\psi], \Sigma_0 \otimes \Sigma_1 + \Sigma_1 \otimes \Sigma_0 \rangle$
- $\langle [1], \Sigma_0 \rangle = h(\mathcal{O}), \quad \langle [\psi], \Sigma_0 \rangle = 0$
- $\langle [\psi], \Sigma_1 \rangle = \log(\mathrm{Red}_N\, u_\psi)$