Origins and applications of higher composition laws

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1801: Gauss' *Disquisitiones*

Let $D \equiv 0, 1 \pmod 4$ be negative.

Theorem (Reduction algorithm)

Every positive definite primitive binary quadratic form is $\mathrm{SL}_2(\mathbf{Z})$*-equivalent to a unique form* $ax^2 + bxy + cy^2$ *with*

$$|b| \leq a \leq c, \qquad \text{and } b \geq 0 \text{ whenever } b = a \text{ or } a = c.$$

This gives a fundamental domain for the action of $\mathrm{SL}_2(\mathbf{Z})$ on the space of (integral) binary quadratic forms (cf. fundamental domain for the action of the modular group on the upper half plane).

Corollary

The number of $\mathrm{SL}_2(\mathbf{Z})$*-classes of positive definite primitive binary quadratic forms of discriminant* D *is finite.*

Question: if n and m are two integers represented by binary quadratic forms f and g, is the product nm represented by a binary quadratic form?

This leads naturally to Gauss composition: if the answer is yes and nm is represented by h, then there are two integral bilinear forms α, β such that

$$f(x,y)g(z,w) = h\left(\alpha\left(x,y,z,w\right), \beta\left(x,y,z,w\right)\right);$$

h is said to be *composed of f and g*.

Theorem (Gauss)

Composition induces a group law on the set of $\mathrm{SL}_2(\mathbf{Z})$-equivalence classes of primitive positive definite binary quadratic forms of discriminant D.

(Note: the notion of group and group action were introduced later on)

Later the same century: Dedekind (?)

Let $\mathcal{O}$ be the (imaginary) quadratic order of discriminant D.

Theorem

There is an isomorphism between the group of classes of primitive positive definite binary quadratic forms and the ideal class group of $\mathcal{O}$.

One can define such an isomorphism by:

$$ax^2 + bxy + cy^2 \longmapsto \left[a, \frac{-b+\sqrt{D}}{2}\right]_{\mathbf{Z}} \subset \mathcal{O}.$$

This proves the finiteness of class numbers of quadratic orders.

Gauss composition can be used to make computations with ideal classes of quadratic orders very explicit (cf. *NUCOMP* algorithm).

2001: Bhargava's Odissey

Where would one go and look for generalizations of Gauss composition?

We have: $G \circlearrowright V$.

$G = \mathrm{SL}_2$, V the space of forms.

First idea: look for pairs (G, V), defined over $\mathbf{Z}$ and parametrizing *interesting* objects.

Notice:

The action of $\mathrm{GL}_2(\mathbf{C})$ on the space of binary quadratic forms over $\mathbf{C}$ *essentially* has one orbit: any two forms with nonzero discriminant can be mapped to one another by a transformation in $\mathrm{GL}_2(\mathbf{C})$.

In other words, over $\mathbf{C}$ there is only one pair (S, I) with S a nondegenerate quadratic ring and I an oriented ideal class of S: $S = I = \mathbf{C} \oplus \mathbf{C}$.

Prehomogeneous vector spaces

Definition

A prehomogeneous vector space is a pair (G, V) where G is an algebraic group and V a rational vector space representation of F such that $G(\mathbf{C}) \circlearrowright V(\mathbf{C})$ has a Zariski-dense orbit.

Composition laws describing orders and ideals in number fields must come from $G(\mathbf{Z}) \circlearrowright V(\mathbf{Z})$, for prehomogeneous spaces defined over $\mathbf{Z}$.

Sato–Kimura, 1977: classification of complex reduced, irreducible prehomogeneous spaces in 36 types.

Wright–Yukie, 1992: over a field K, K-orbits of prehomogeneous spaces *often* correspond to field extensions of K.

V	G	Parametrizes
$D \equiv 0, 1 \pmod 4$	$\mathrm{SL}_1(\mathbf{Z})$	Quadratic rings
$(\mathrm{Sym}^2\mathbf{Z}^2)^*$	$\mathrm{SL}_2(\mathbf{Z})$	Ideal classes in quadratic rings
$\mathrm{Sym}^3\mathbf{Z}^2$	$\mathrm{SL}_2(\mathbf{Z})$	Order three ideal classes in quad. rgs.
$\mathbf{Z}^2 \otimes \mathrm{Sym}^2\mathbf{Z}^2$	$\mathrm{SL}_2(\mathbf{Z})^2$	Ideal classes in quadratic rings
$\mathbf{Z}^2 \otimes \mathbf{Z}^2 \otimes \mathbf{Z}^2$	$\mathrm{SL}_2(\mathbf{Z})^3$	Pairs of ideal classes in quadratic rgs.
$\mathbf{Z}^2 \otimes \wedge^2\mathbf{Z}^4$	$\mathrm{SL}_2(\mathbf{Z}) \times \mathrm{SL}_4(\mathbf{Z})$	Ideal classes in quadratic rings
$\wedge^3\mathbf{Z}^6$	$\mathrm{SL}_6(\mathbf{Z})$	Quadratic rings
$(\mathrm{Sym}^3\mathbf{Z}^2)^*$	$\mathrm{GL}_2(\mathbf{Z})$	Cubic rings
$\mathbf{Z}^2 \otimes \mathrm{Sym}^2\mathbf{Z}^3$	$\mathrm{GL}_2(\mathbf{Z}) \times \mathrm{SL}_3(\mathbf{Z})$	Order two ideal classes in cubic rings
$\mathbf{Z}^2 \otimes \mathbf{Z}^3 \otimes \mathbf{Z}^3$	$\mathrm{GL}_2(\mathbf{Z}) \times \mathrm{SL}_3(\mathbf{Z})^2$	Ideal classes in cubic rings
$\mathbf{Z}^2 \otimes \wedge^2\mathbf{Z}^6$	$\mathrm{GL}_2(\mathbf{Z}) \times \mathrm{SL}_6(\mathbf{Z})$	Cubic rings
$(\mathbf{Z}^2 \otimes \mathrm{Sym}^2\mathbf{Z}^3)^*$	$\mathrm{GL}_2(\mathbf{Z}) \times \mathrm{SL}_3(\mathbf{Z})$	Quartic rings
$\mathbf{Z}^4 \otimes \wedge^2\mathbf{Z}^5$	$\mathrm{GL}_4(\mathbf{Z}) \times \mathrm{SL}_5(\mathbf{Z})$	Quintic rings

Composition laws and exceptional Lie groups

Let G be a Lie group, P a maximal parabolic subgroup of G. We write $P = LU$ where L is the Levi factor and U is the unipotent radical at P.

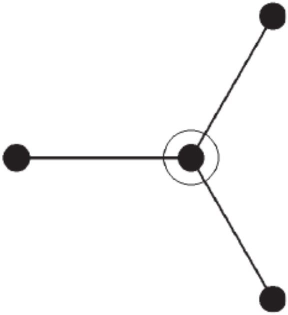
Fact: L acts on $W = U/[U,U]$ by conjugation.
Rubenthaler, Vinberg: pairs (L, W) can be completely classified as prehomogeneous spaces.

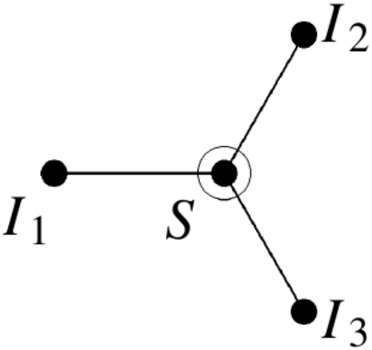
For certain choices of G and P, the group L and space W correspond to the pairs considered in the table.

Example

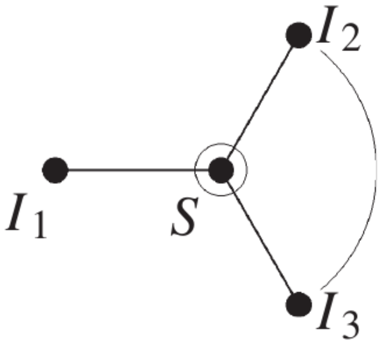
If G is the exceptional Lie group of type D_4 and P corresponds to the central vertex of the associated Dynkin diagram, then $L = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ and W is the space of cubes.

$L = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$
Acts on: (S, I_1, I_2, I_3), S quadratic.
Type: D_4.

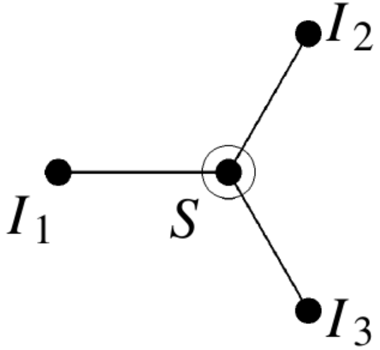


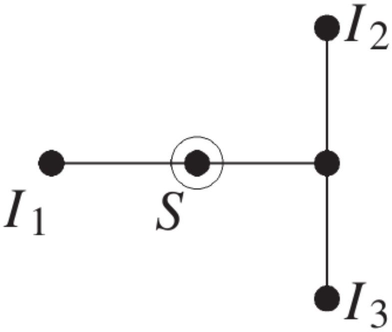


Identify $I_2 = I_3$.
$L = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$
Acts on: (S, I_1, I_2), S quadratic.
Type: B_3

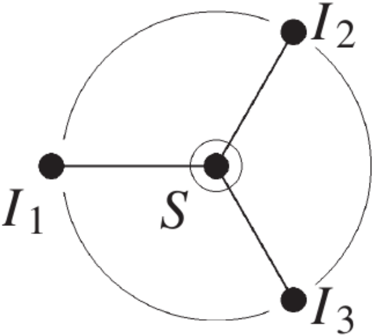


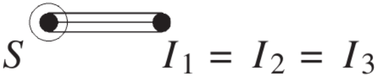
Now instead we fuse I_2 and I_3 by direct sum.
$G = \mathrm{SL}_2 \times \mathrm{SL}_4$
$V = \mathbf{Z}^2 \otimes \wedge^2 \mathbf{Z}^4$
Type: D_5.



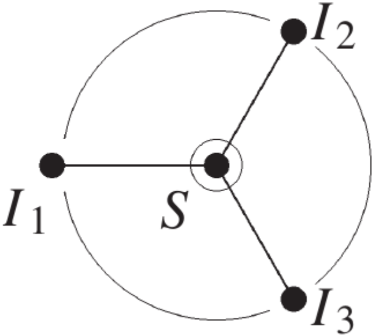


Now identify $I_1 = I_2 = I_3$, so that $I_1^3 \sim S$.
$G = \mathrm{SL}_2$
$V = \mathrm{Sym}^3 \mathbf{Z}^2$
Type: G_2.

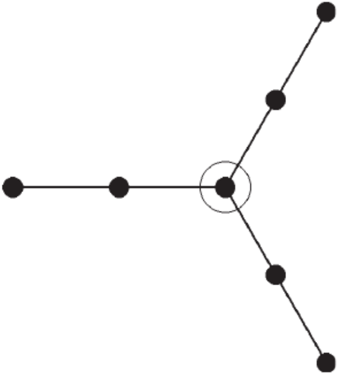




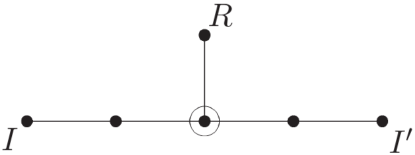
Fuse all three ideals by direct sum.
$G = \mathrm{SL}_6$
$V = \wedge^3 \mathbf{Z}^6$
Type: E_6.



What about the cubic case? Why is there no relevant composition law for 3x3x3 cubes of integers? Because it would take a Dynkin diagram of the form:

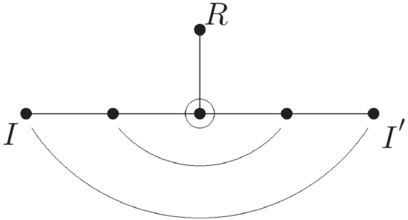


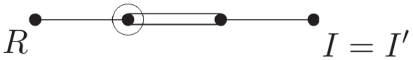
Instead, let's cut one of the legs short and consider 2x3x3 cubes of integers:



R a cubic ring, I, I' a couple of (balanced ideals).
$\Gamma = \mathrm{GL}_2 \times \mathrm{GL}_3 \times \mathrm{GL}_3$.
$V = \mathbf{Z}^2 \otimes \mathbf{Z}^3 \otimes \mathbf{Z}^3$.
Type: E_6.

Now identify $I = I'$, which implies in particular $I^2 \sim R$.





R a cubic ring, I, I' a couple of (balanced ideals).
$\Gamma = \mathrm{GL}_2 \times \mathrm{GL}_3$.
$V = \mathbf{Z}^2 \otimes \mathrm{Sym}^2\mathbf{Z}^3$.
Type: F_4.

V	G	Type
$D \equiv 0, 1 \pmod 4$	$\mathrm{SL}_1(\mathbf{Z})$	A_1
$(\mathrm{Sym}^2 \mathbf{Z}^2)^*$	$\mathrm{SL}_2(\mathbf{Z})$	B_2
$\mathrm{Sym}^3 \mathbf{Z}^2$	$\mathrm{SL}_2(\mathbf{Z})$	G_2
$\mathbf{Z}^2 \otimes \mathrm{Sym}^2 \mathbf{Z}^2$	$\mathrm{SL}_2(\mathbf{Z})^2$	B_3
$\mathbf{Z}^2 \otimes \mathbf{Z}^2 \otimes \mathbf{Z}^2$	$\mathrm{SL}_2(\mathbf{Z})^3$	D_4
$\mathbf{Z}^2 \otimes \wedge^2 \mathbf{Z}^4$	$\mathrm{SL}_2(\mathbf{Z}) \times \mathrm{SL}_4(\mathbf{Z})$	D_5
$\wedge^3 \mathbf{Z}^6$	$\mathrm{SL}_6(\mathbf{Z})$	E_6
$(\mathrm{Sym}^3 \mathbf{Z}^2)^*$	$\mathrm{GL}_2(\mathbf{Z})$	G_2
$\mathbf{Z}^2 \otimes \mathrm{Sym}^2 \mathbf{Z}^3$	$\mathrm{GL}_2(\mathbf{Z}) \times \mathrm{SL}_3(\mathbf{Z})$	F_4
$\mathbf{Z}^2 \otimes \mathbf{Z}^3 \otimes \mathbf{Z}^3$	$\mathrm{GL}_2(\mathbf{Z}) \times \mathrm{SL}_3(\mathbf{Z})^2$	E_6
$\mathbf{Z}^2 \otimes \wedge^2 \mathbf{Z}^6$	$\mathrm{GL}_2(\mathbf{Z}) \times \mathrm{SL}_6(\mathbf{Z})$	E_7
$(\mathbf{Z}^2 \otimes \mathrm{Sym}^2 \mathbf{Z}^3)^*$	$\mathrm{GL}_2(\mathbf{Z}) \times \mathrm{SL}_3(\mathbf{Z})$	F_4
$\mathbf{Z}^4 \otimes \wedge^2 \mathbf{Z}^5$	$\mathrm{GL}_4(\mathbf{Z}) \times \mathrm{SL}_5(\mathbf{Z})$	E_8

Density of discriminants

Let R be a ring admitting a **Z**-basis $\{\alpha_1, \dots, \alpha_n\}$.

Definition

$\mathrm{Disc}(R) = \det\left(\mathrm{Tr}(\alpha_i \alpha_j)\right)_{1 \le i,j \le n}$.

For a number field K, $\mathrm{Disc}(K) = \mathrm{Disc}(\mathcal{O}_K)$.

Theorem (Minkowski)

For number fields, up to isomorphism, $\#\{K;\ \mathrm{Disc}(K) = D\} < \infty$.

We set $N_n(x) = \#\{K;\ \mathrm{Gal}(K^g|\mathbf{Q}) = \mathfrak{S}_n,\ \mathrm{Disc}(K) \le x\}$.

Conjecture

The limit

$$c_n = \lim_{x \to +\infty} \frac{N_n(x)}{x}$$

exists and is positive for all $n \ge 2$.

Example

$c_1 = 0,\ c_2 = 6/\pi^2 = 1/\zeta(2).$

For c_2: we need to count integers up to x which are 0 or 1 $\pmod 4$ and squarefree.

Davenport–Heilbronn, 1971: $c_3 = 1/3\zeta(3)$
(via **Delone–Faddeev, 1964**).

Bhargava, 2005 and **2010**: values of c_4 and c_5.

$n = 3$. Davenport–Heilbronn

Cubic orders are parametrized by $\mathrm{GL}_2(\mathbf{Z})\backslash(\mathrm{Sym}^3\mathbf{Z}^2)^*$, that is: $\mathrm{GL}_2(\mathbf{Z})$-equivalence classes of binary cubic forms.
They form a lattice in the 4-dimensional $\mathbf{R}$-vector space

$$V = (\mathrm{Sym}^3\mathbf{R}^2)^* = \left\{ax^3 + bx^2y + cxy^2 + dy^3;\ a, b, c, d \in \mathbf{R}\right\}.$$

Davenport–Heilbronn: explicitly construct a fundamental domain $\mathcal{F}$ for $\mathrm{GL}_2(\mathbf{Z}) \circlearrowright V$.

The number of cubic orders having discriminant at most x is the number of integer points in the region

$$\mathcal{F}_x = \mathcal{F} \cap \{v \in V;\ |\mathrm{Disc}(v)| \le x\}$$

Toy example

For counting SL_2-classes of positive definite primitive binary quadratic forms of discriminant at most x, we would take
$V = \left\{ax^2 + bxy + cy^2;\ a, b, c \in \mathbf{R}\right\}$, $\Gamma = \mathrm{SL}_2(\mathbf{Z})$,

$$\mathcal{F} = \{|b| < a < c\} \cup \{0 < b < a = c\} \cup \{0 < b = a < c\}$$

$$\mathcal{F}_x = \mathcal{F} \cap \left\{|b^2 - 4ac| \leq x\right\}$$

The number of integral points in a region $\mathcal{R}$ in Euclidean space can be approximated correctly by its volume provided:

1. $\mathcal{R}$ is compact.
2. $\mathcal{R}$ is *round-looking* (smooth boundaries and no serious *spikes* or *tentacles*).

Fact: $\mathrm{vol}(\mathcal{F}_x) = \pi^2/18$.

Problem: $\mathcal{F}_x$ has a tentacle going to infinity, arising from non-compactness of $\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R})$.

Davenport–Heilbronn: most points in the tentacle correspond to reducible binary cubic forms (ie: a binary quadratic form times a linear form). The number of integral points in the tentacle corresponding to irreducible forms is $O(x)$.
Similarly, the contrary happens in the *smooth* part of $\mathcal{F}_x$.

Theorem

$$\#\{\ \textit{cubic orders } R;\ \mathrm{Disc}(R) \leq x\} \sim \frac{\pi^2}{18}x, \quad (x \to \infty).$$

Are we done?

NO: passing from cubic orders to maximal cubic orders requires a rather delicate sieve.

Theorem

$$\#\{K;\ [K:\mathbf{Q}] = 3, \mathrm{Disc}(K) \leq x\} \sim \frac{1}{3\zeta(3)}x, \quad (x \to \infty).$$

$n = 4$. Bhargava

Same ideas with much more difficult computations and subtler obstacles. We have V, G, $\mathcal{F}$, $\mathcal{F}_x$ as above, but now:

$$V = (\mathbf{R}^2 \otimes \mathrm{Sym}^3 \mathbf{R}^2)^*, \quad G = \mathrm{GL}_2(\mathbf{Z}) \times \mathrm{SL}_3(\mathbf{Z}).$$

Now $\dim_{\mathbf{R}} V = 12$. Bhargava constructs $\mathcal{F}$ and computes:

$$\mathrm{vol}(\mathcal{F}_x) = \frac{5}{24}\zeta(2)^2\zeta(3)x.$$

Problem: $\mathcal{F}_x$ has three big *cusps*.

1st cusp: reducible points corresponding to $Q = S_1 \oplus S_2$, S_i quadratic.
2nd cusp: $Q = R \oplus L$, R cubic and L linear.
3rd cusp: irreducible points corresponding to D_4-quartic fields.

Theorem

Let $\Xi(x)$ be the set of isomorphism classes of pairs (Q, R), where Q is an $\mathfrak{S}_4$-quartic order of discriminant at most x and R is a cubic resolvent of Q. Then:

$$\#\Xi(x) \sim \frac{5}{24}\zeta(2)^2\zeta(3)x, \quad (x \to \infty)$$

We need to drop the R in order to count only isomorphism classes of Q.

Theorem

Let $\Theta(x)$ be the set of isomorphism classes of $\mathfrak{S}_4$-quartic orders with discriminant at most x.

$$\#\Theta(x) \sim \frac{5}{24}\frac{\zeta(2)^2\zeta(3)}{\zeta(5)}x, \quad (x \to \infty).$$

Going from orders to maximal orders requires again a sieve (**hard!**).

Theorem

$$c_4 = \frac{5}{24} \prod_p (1 + p^{-2} - p^{-3} - p^{-4}).$$

Corollary

When ordered by size of discriminant, quartic fields are:

- 90.644%: *of $\mathfrak{S}_4$-type.*
- *The rest: of D_4-type.*
- 0%: *other Galois groups.*

(By Hilbert irreducibility, if we order degree n polynomials by size of coefficients, 100% *are of S_4-type).*

$n = 5$. Bhargava

$$V = \mathbf{R}^4 \otimes \wedge^2 \mathbf{R}^5, \quad G = \mathrm{GL}_4(\mathbf{Z}) \times \mathrm{SL}_5(\mathbf{Z}).$$

Now $\dim_{\mathbf{R}}(V) = 40$.
Bhargava constructs $\mathcal{F}$ and computes the volume of $\mathcal{F}_x$.

Problem: $\mathcal{F}_x$ is highly non-compact.

- There are 160 cusps. They contain points that can be discarded (reducible, other Galois types).
- 100% of integral points corresponding to orders in $\mathfrak{S}_5$-quintic fields are away from the cusps.

Theorem

$$c_5 = \frac{13}{120} \prod_p \left(1 + p^{-2} - p^{-4} - p^{-5}\right).$$

Getting started

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Curiosities

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Thanks