On the automorphisms groups of genus 3 curves

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Notation

C non-singular, projective curve over K,
K algebraic closed field, $char(K) = 0$.
Impose genus(C) ≥ 2, and denote it by g.
$WP(C)$ the set of all Weierstrass points of C
Remember

$$2g + 2 \leq \#W(C) \leq (g-1)g(g+1),$$

and $\#W(C) = 2g + 2$ if and only if C is hyper-elliptic.
WP denotes a single Weierstrass point of C.

$Aut(C)$ group of K-automorphism of the curve C.
$v(\varphi)$ the number of fixed points of φ.

Consider a separable covering:

$$\pi : C \to C'$$

with g' is genus C', Remember Hurwicz formula:

$$2g - 2 = deg(\pi)(2g' - 2) + \sum_{P \in C} (e_P - 1) =$$

$$deg(\pi)(2g' - 2) + \sum_{i=1}^{r} \frac{deg(\pi)}{v_j}(v_j - 1)$$

$$= deg(\pi)(2g' - 2) + deg(\pi) \sum_{i=1}^{r} (1 - v_j^{-1}),$$

where r are the points on C' over which the ramification occurs $\tilde{P}_1, \ldots, \tilde{P}_r$, and for each $\tilde{P}_j$ there are $deg(\pi)/v_j$ branch points in C.
$P_j^1, \ldots, P_j^{deg(\pi)/v_j}$ each with ramification $v_j = e_{P_j^l}$.

1.General facts on the group $Aut(C)$

Lemma 1. *Let φ be an element of $Aut(C)$ different from the identity. Then φ fixes at most 2g+2 points (i.e. $v(\varphi) \leq 2g+2$).*

Proof. S set of fixed points of φ.
$P \in C$ non-fixed by φ.
exist f with $(f)_\infty = rP$ (the poles of f) for some r with $1 \leq r \leq g+1$ (take $r = g+1$ if P is not a WP).
$h := f - f\varphi$, then $(h)_\infty = rP + r(\varphi^{-1}P)$, thus h has $2r(\leq 2g+2)$ zeroes. Observe that every fixed point of φ is a zero for h. □

Lemma 2. *Let be $\varphi \in Aut(C)$. If P is a WP of C then $\varphi(P)$ is a WP of C.*

$S_{WP(C)}$ the permutation group on the set of Weierstrass points. We have:

$$\lambda : Aut(C) \to S_{WP(C)}.$$

Lemma 3. *λ is injective unless C is hyperelliptic. If C is hyperelliptic, then $ker(\lambda) = \{id, w\}$ where w denotes the hyperelliptic involution on C.*

Proof. Let be $\phi \in ker(\lambda)$. If C is non-hyperelliptic, we have strictly more than $2g + 2$ WP points, thus by lemma 1 $\phi = id$.

□

If C is non-hyperelliptic we have a canonical immersion

$$\phi : C \to \mathbb{P}^{g-1},$$

and a canonical model of C in the projective $\phi(C)$.

Proposition 4. *If C is a non-hyperelliptic curve, then we can think any $Aut(C)$ as a projective transformation on $\mathbb{P}^{g-1}$ leaving $\phi(C)$ invariant.*

Hurwicz formula consequences:

Lemma 5. *Let be $\varphi \in Aut(C)$ of prime order p. Then $p \leq g$ or $p = g+1$ or $p = 2g+1$.*

Proof. Let us apply the Hurwicz formula in the Galois covering $\pi : C \to C/ < \varphi >$. Denote by $\tilde{g}$ the genus of $C/ < \varphi >$, Hurwicz formula reads:

$$2g - 2 = p(2\tilde{g} - 2) + v(\varphi)(p - 1).$$

Assume once and for all in this proof $p \geq g+1$. If $\tilde{g} \geq 2$ then we have $2g-2 \geq p(2\tilde{g}-2) \geq 2p \geq 2g+2$, this can not happen.

If $\tilde{g} = 1$ then we have $2g - 2 = v(\varphi)(p - 1) \geq v(\varphi)g$ if $v(\varphi) \geq 2$ this can not happen. It can not happen also that $v(\varphi) = 1$ (any automorphism of prime order of $Aut(C)$ which has one fixed point, it must have at least two).

If $\tilde{g} = 0$ then if $v(\varphi) \geq 5$ we have $2g - 2 = -2p + v(\varphi)(p - 1) \geq 3p - 5 \geq 3g - 2$, this can not happen. If $v(\varphi) = 4$ then by Hurwicz $2g-2 = -2p+4(p-1) = 2p-4$ this can only happen for $g = p + 1$. If $v(\varphi) = 3$ then $2g - 2 = -2p + 3(p - 1) = p - 3$ which can only happen with $p = 2g + 1$. $\square$

Similar arguments from the Hurwicz formula as above lemma prove the following results;

Theorem 6 (Hurwicz, 1893)**.** *For any C with genus $g \geq 2$ we have that*

$$\# Aut(C) \leq 84(g - 1).$$

Here we use the cover $C \to C/Aut(C)$ and apply Hurwicz formula.

Proposition 7 (Hurwicz, 1893). *Let be H a cyclic subgroup of $Aut(C)$ and denote by $\tilde{g}$ the genus of C/H and $m = \#H$. Then:*

1. *if $\tilde{g} \geq 2$ then $m \leq g - 1$.*

2. *if $\tilde{g} = 1$ then $m \leq 2(g - 1)$.*

3. *if $\tilde{g} = 0$ and* $\begin{cases} r \geq 4 \Rightarrow m \leq 2(g-1). \\ r = 4 \Rightarrow m \leq 6(g-1). \\ r = 3 \Rightarrow m \leq 10(g-1). \end{cases}$

The proof is similar to Hurwicz theorem with the cover $\pi : C \to C/H$.

Let us finally list some other properties which follows from Hurwicz formula :

Proposition 8 (Accola). *Let be H and H_j $1 \leq j \leq k$ subgroups of $Aut(C)$ such that $H = \bigcup_{j=1}^{k} H_j$ and $H_i \cap H_l = \{id\}$ with $i \neq l$. Denote by $m_j = \#H_j$ and $m = \#H$ and $\tilde{g}$ the genus of C/H and $\tilde{g}_j$ the genus of C/H_j. Then,*

$$(k-1)g + m\tilde{g} = \sum_{j=1}^{k} m_j \tilde{g}_j.$$

Corollary 9. *Let C be a genus 3 curve which is non-hyperelliptic. Then any involution σ on C is a 2-hyperelliptic that means a bielliptic involution (i.e. the genus of $C/ < \sigma >$ is 1).*

Corollary 10. *If C has genus 3 and suppose that exist a subgroup H of $Aut(C)$ isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ such that the genus of C/H is zero. If one element of H fix no point of C then C is an hyperelliptic curve.*

Lemma 11. *Let be $\varphi \in Aut(C)$ not the identity. Let be $v(\varphi)$ the number of fixed points. Then $v(\varphi) \leq 2 + \frac{2g}{ord(\varphi)-1}$ where $ord(\varphi)$ is the order of this element in the group.*

Proposition 12. *Let be $\varphi \in Aut(C)$ not the identity. If $v(\varphi) > 4$ then every fixed point of φ is a WP.*

Let us make explicit general facts on $Aut(C)$ when the genus C is three:

$$C,\ with\ genus(C) = 3, then:$$

$$\#Aut(C) \leq 168$$

Only the primes $2, 3, 7$ can divide the order of $Aut(C)$

$Aut(C)$ is a finite group in $PGL_3(K)$

Automorphism groups appearing on genus 3 curves

C once and for all a non-hyperelliptic genus 3 curve.
Think C embedded in $\mathbb{P}^2$, its equation is a non-singular plane quartic in $\mathbb{P}^2$(with at least degree 3 in every variable).

This talk follows two approaches to list the full list of groups appearing:

-Komiya-Kubayashi (Section 2.2)
-Dolgachev (Section 2.1)

Both approaches study first a cyclic subgroup of $Aut(C)$ in order to obtain a model equation, and latter from this equation obtain the fuller automorphism group.

2.1. The determination of the finite groups inside PGL_3

Take every finite group inside PGL_3 with less than 168 elements and only has prime orders 2,3 or 7.
Study which has as fixed set of $(X : Y : Z)$ a non-singular plane quartic.

Proposition 13. *Let g be an automorphism of order $m > 1$ of a non-singular quartic plane curve $C = V(F(X, Y, Z))$. Let us choose coordinates in such a way such that generator of the cyclic group $H =< g >$ is represented by a diagonal matrix*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi_m^a & 0 \\ 0 & 0 & \xi_m^b \end{pmatrix},$$

where ξ_m is a primitive $m-$th root of unity. Then $F(X, Y, Z)$ is in the following list:

Cyclic automorphism of order m.

$g = diag[1, \xi_m^a, \xi_m^b]$ *we denote its Type by:* $m, (a, b)$

$C = V(F)$ *where* C *denotes the quartic.*

L_i *denotes homogenous polynomial of degree* i

	Type	$F(X,Y$
(i)	$2,(0,1)$	$Z^4+Z^2L_2(X,Y)+L_4(X$
(ii)	$3,(0,1)$	$Z^3L_1(X,Y)+L_4(X$
(iii)	$3,(1,2)$	$X^4+\alpha X^2YZ+XY^3+XZ^3+\beta Y$
(iv)	$4,(0,1)$	$Z^4+L_4(X$
(v)	$4,(1,2)$	$X^4+Y^4+Z^4+\delta X^2Z^2+\gamma X$
(vi)	$6,(3,2)$	$X^4+Y^4+\alpha X^2Y^2+$
(vii)	$7,(3,1)$	X^3Y+Y^3Z+
(viii)	$8,(3,7)$	X^4+Y^3Z+
(ix)	$9,(3,2)$	X^4+XY^3+
(x)	$12,(3,4)$	X^4+Y^4+

Proof. (sketch)
g acts by $(X : Y : Z) \mapsto (X : \xi_m^a Y : \xi_m^b Z)$.
First situation: Suppose or a or b is zero.
Assume $a = 0$, write

$$F = \beta Z^4 + Z^3 L_1(X,Y) + Z^2 L_2(X,Y) +$$

$$Z L_3(X,Y) + L_4(X,Y),$$

if $\beta \neq 0$
then $4b \equiv 0 \bmod m$, thus $m = 2$ or $m = 4$.
If $m = 2$ then $L_1 = L_3 = 0$ we obtain Type $2, (0,1)$.
If $m = 4$ ($b \neq 2$), then $L_1 = L_2 = L_3 = 0$ and we get Type $4, (0,1)$.
If $\beta = 0$, then $3b = 0$ mod m, then $m = 3$ and thus $L_2 = L_3 = 0$ and we get Type $3, (0,1)$.
Second situation: $0, a, b$ are all distint. We can suppose that $a \neq b$, $(a,b) = 1$.Let be $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$ and $P_3 = (0 : 0 : 1)$ the reference points.

1. All reference points lie in the non-singular plane quartic.
   The equation possibilities are now:

   $$F = X^3L_{1,X}(Y,Z)+Y^3L_{1,Y}(X,Z)+Z^3L_{1,Z}(X,Y)$$

   $$+X^2L_{2,X}(Y,Z)+Y^2L_{2,Y}(X,Z)+Z^2L_{2,Z}(X,Y)$$

   By change on the variables X,Y,Z, reduces to:

   $$F = X^3Y + Y^3Z + Z^3X + X^2L_{2,X}(Y,Z)+$$
   $$Y^2L_{2,Y}(X,Z) + Z^2L_{2,Z}(X,Y).$$

   We have that $a = 3a + b = 3b \bmod m$, therefore $m = 7$ and take a generator of H such that $(a,b) = (3,1)$. We obtain that no other monomial enters in F then Type $7,(3,1)$.

2. Two reference points lie in the plane quartic.

   By rescalaring the matrix g joint with permuting the coordinates we can assume that $(1:0:0)$ does not lie in C. The equation is like:

   $$F = X^4 + X^2 L_2(Y,Z) + X L_3(Y,Z) + L_4(Y,Z)$$

   because as $a, b \neq 0$ L_1 does not appear (is not invariant by g), moreover Y^4 and Z^4 are not in L_4 (only $(1:0:0)$ does not lie in C).

   First: Y^3Z appears in L_4. We have $3a+b = 0 \bmod m$.

   Suppose Z^3Y is also in L_4 then $a + 3b = 0$ therefore $8b = 0 \bmod m$ and then $m = 8$, we can take g generator with $(a,b) = (3,7)$ and we obtain Type $8, (3,7)$.

   If Z^3Y is not in L_4 then Z^3 is in L_3 (because non-singularity) therefore $3b = 0 \bmod m$, this condition joint with 3a+b=0 mod$m$

we have two situations: $m = 3$ and take g with $(a, b) = (1, 2)$ or $m = 9$ and $(a, b) = (3, 2)$, but the first can not happen under the condition that Y^3Z appears in L_4 and the second type is equal to $9, (3, 2)$ of the table.
Second: If Y^3Z is not but is Z^3Y in L_4 a permutation of $Y \leftrightarrow Z$ we reduce above situation. If Y^3Z and Z^3Y are not in L_4 for non-singularity we have that Y^3 and Z^3 should appear in L_3, then $3b = 0$ and $3a = 0$ mod m, therefore $m = 3$ and $(a, b) = (1, 2)$ is the Type $3, (1, 2)$ in the table.

3. One reference point lie in the plane quartic. We assume that $P_1 = (1 : 0 : 0)$ and $P_2 = (0 : 1 : 0)$ do not lie on C.

$$F = X^4 + Y^4 + X^2 L_2(Y, Z) + X L_3(Y, Z) + L_4(Y, Z).$$

4. None of the reference points lies in the plane quartic.
   In this situation

$$F = X^4 + Y^4 + Z^4 + X^2 L_2(Y, Z) + X L_3(Y, Z) +$$

$$\alpha Y^3 Z + \beta Y Z^3 + \iota Y^2 Z^2,$$

5. One reference point lie in the plane quartic. We assume that $P_1 = (1:0:0)$ and $P_2 = (0:1:0)$ do not lie on C.

$F = X^4+Y^4+X^2L_2(Y,Z)+XL_3(Y,Z)+L_4(Y,Z)$,

where Z^4 does not enter in L_4 for the hypotheses on which references points lie or not lie in the quartic, L_1 does not appear because $ab \neq 0$. We have then $4a = 0 \mod m$. By non-singularity Z^3 appears in L_3, therefore $3b = 0 \mod m$, hence $m = 6$ or $m = 12$. Imposing the invariance by g we obtain

$$(*)F = X^4 + Y^4 + \alpha X^2Y^2 + XZ^3,$$

if $m = 6$ then $(a,b) = (3,2)$ (and α may be diferent from 0), this is Type $6,(3,2)$. If $m = 12$ then $(a,b) = (3,4)$ from the above equation $(*)$ and $\alpha = 0$, this is Type $12,(3,4)$.

6. None of the reference points lies in the plane quartic.
   In this situation

$$F = X^4 + Y^4 + Z^4 + X^2 L_2(Y,Z) + X L_3(Y,Z) +$$

$$\alpha Y^3 Z + \beta Y Z^3 + \iota Y^2 Z^2,$$

   where L_1 does not appears because $ab \neq 0$. Clearly $4a = 4b = 0 \mod m$, therefore $m = 4$ and we can take $(a,b) = (1,2)$ or $(1,3)$ both situation define isomorphic curves (only by a renaming which is X, Y, Z in the equations), this is type $4, (1,2)$.

□

Group theory notation

$G \leq GL(V)$. G intransitive if the representation of G in $GL(V)$ is reducible. Otherwise we say transitive.
G is imprimitive if G contains an intransitive normal subgroup G',
in this situation V decomposes into direct sum of G'-invariant proper subspaces and the set of representants of G of G/G' permutates them.

C_m the cyclic group of order m,
S_i the simetric group of i-elements,
A_i the alternate group of i-elements,
D_i the dihedral group, order $2i$.
Q_8 the quaternion group
$A \circledcirc B$ denotes a group G defined as an $Ext^1(B, A)$, from

$$1 \to A \to G \to B \to 1,$$

and $A \rtimes B$ the semi-product.

Theorem 14. *In the following table we list all the groups that appear as a group of automorphism of a non-singular plane quartic and moreover group by group we list equations which has exactly as automorphism group this group. These equations covers up to isomorphism all the plane non-singular quartics which has some automorphism.*

Full automorphism group G.

n denotes order of the group G

n	G	
168	$PSL_2(\mathbb{F}_7) \cong PSL_3(\mathbb{F}_2)$	
96	$(C_4 \times C_4) \rtimes S_3$	
48	$C_4 \circledcirc A_4$	
24	S_4	$Z^4 + Y^4 + X^4 + 3a(Z$ …
16	$C_4 \circledcirc (C_2 \times C_2)$	$Z^4 - X^3Y +$ …
9	C_9	
8	Q_8	$Z^4 + \alpha Z^2(Y^2 + X^2$ …
6	C_6	Z^4 …
6	S_3	$Z^4 + \alpha Z^2YX +$ …
4	$C_2 \times C_2$	$Z^4 + Z^2(\alpha Y^2 + \beta X^2$ …
3	C_3	
2	C_2	$Z^4 +$ …

where $P.M.$ means parameter restriction.

Remark 15. *the Dolgachev table is:*

n	G	
168	$PSL_2(\mathbb{F}_7) \cong PSL_3(\mathbb{F}_2)$	
96	$(C_4 \times C_4) \rtimes S_3$	
48	$C_4 \circledcirc A_4$	
24	S_4	$Z^4 + Y^4 + X^4 + a(Z$
16	$C_4 \times C_4$	$Z^4 +$
9	C_9	
8	Q_8	$Z^4 + \alpha Z^2(Y^2 + X^2)$
7	C_7	$Z^3Y +$
6	C_6	Z^4
6	S_3	$Z^4 + \alpha Z^2YX +$
4	$C_2 \times C_2$	$Z^4 + Z^2(\alpha Y^2 + \beta X^2)$
3	C_3	$Z^4 + \alpha Z^2YX +$
2	C_2	$Z^4 + Z$

Proof. (sketch)
Case 1: G an intransitive group realized as a group of automorphism.
Case 1.a.:$V = V_1 \oplus V_2 \oplus V_3$.
Choose (X, Y, Z) such that V_1 spanned by $(1, 0, 0)$ and so on.
$g \in G$ of order m, after scaling $g = diag(1, a, b)$, we know models of equations and restrictions for m, a, b above proposition.
Suppose $h \in G$ but $h \notin < g >$, (choose m maximal with the property that has an element of order m).
Study now situation by situation the equations on cyclic subgroups (i)-(x):
Take $m = 12$, (x); we think $h = diag(1, \xi^c_{m'}, \xi^d_{m'})$ then $4c = 3d = 0 \mod m'$, then $12|m'$ and $h \in < g >$.
Nevertheless situation (x) has bigger automorphisms group which appears in case 1.b.
Similar arguments in the cases (v)-(x) to conclude: there are no other automorphism appearing as an intransitive group with $V = V_1 \oplus$

$V_2 \oplus V_3$.
Case (iv) and suppose $h \notin < g >$, let

$$L_4 = aX^4 + bY^4 + cX^3Y + dXY^3 + eX^2Y^2$$

assume $ab \neq 0$, $h = diag(\xi_{m'}^p, \xi_{m'}^q, 1)$, then $m' = 2$ or 4. If $m' = 2$ the only possibility is $(p, q) = (0, 1)$ or $(1, 0)(h \notin < g >)$ where $c = d = 0$, but in this possibility we obtain a bigger group of automorphism.
If $m' = 4$, only possibilities $(p, q) = (1, 0)$,$(0, 1)$, $(1, 3)$, $(3, 1)$, $(1, 2)$,$(2, 1)$. If $(p, q) = (1, 3)$ or $(3, 1)$ we have $c = d = 0$, we obtain that this equation that has bigger group appearing in the following process (interchanging X and Y). If (p, q)= $(1, 2)$ or $(2, 1)$ similar as the case $(1, 3)$. The situation$(1, 0)$ implies $c = d = e = 0$, this is the Fermat quartic and has bigger group of automorphism.
Assume now $a \neq 0$ and $b = 0$. $d \neq 0$ (non-singularity). One has $4p = 3p + q = 0$ mod m', then $c = e = 0$. But then we obtain $\mathbb{Z}/12$ as

group, situation (x) considered before.
Assume now $a = b = 0$. $cd \neq 0$(non-singularity). $3p+q = p+3q = 0$ mod (m'), but then $m' = 8$ (studied above).
Similar argument applied:
Case (iii) One checks that no other element appears except when 1)$\alpha = \beta = 0$ that is situation (ix) already studied;2)$\alpha = \beta$ appears C_6 in the group an is already studied (vi),3)$\beta = 0$,$\alpha \neq 0$ no-reduced,4)$\alpha = 0$, $\beta \neq 0$ appears C_6.
Case (ii): Since $L_1 \neq 0$ no h can appear.
Case (i): Only need to study when $diag(1, -1, 1)$ appears (i.e. we have $C_2 \times C_2$). We have that L_4 does not contain Y^3X and X^3Y and L_2 does not contain XY. In this situation could has a bigger group of automorphism when $\alpha = \beta$ (see table).

Case 1.b. $V = V_1 \oplus V_2$ with dimV_2=2, where V_2 irreducible representation of G (G non-abelian).

Choose coordinates s.t. $(1,0,0) \in V_1$, V_2 spanned by $(0,1,0),(0,0,1)$. $\overline{g}$ restriction of g to $W = V(Z) = \mathbb{P}(V_2)$, choose in SL_2. Write:

$$F = \alpha Z^4 + Z^3 L_1(Y,X) + Z^2 L_2(Y,X) + Z L_3(Y,X)$$

$$+ L_4(Y,X),$$

$L_1 = 0$ (irreducibility of V_2) and $\alpha \neq 0$(non-singularity).

If $L_2 \neq 0$, G leaves $V(L_2)$ invariant, $\overline{G}$ the restriction of G in W, the

$$G \leq D_2$$

then by a change of variables on V_2 that the action of $\overline{G}$ is $(x,y) \mapsto (-y,x)$ and $(x,y) \mapsto (ix,-iy)$, then G can be only an extension of the $C_2 \times C_2$ situation above, therefore we have that G is isomorphic to Q_8 with the values on the table of the theorem.

If $L_2 = 0$ but $L_3 \neq 0$, here $\overline{G} \leq D_3$ obtains that with the invariants of this elements one obtains a singular curve.

If $L_2 = L_3 = 0$ but $L_4 \neq 0$, $\overline{G}$ leave $V(L_4)$ invariant. One knows

$$\overline{G} \leq A_4$$

of order 12. One should study all these subgroups, the ones with has $\mathbb{Z}/2 \times \mathbb{Z}/2$ we can restrict on the equation given by step 1a and one obtains the group of 16 elements and the group of 48 elements.

Case 2: G has normal transitive imprimitive subgroup H.
H is a subgroup given above and permutates cyclically coordinates, therefore the only situations possible are:

$$Z^4 + \alpha Z^2 Y X + Z(Y^3 + X^3) + \beta Y^2 X^2$$

$$Z^3 Y + Y^3 X + X^3 Z$$

$$Z^4 + Y^3X + X^3Y$$

$$Z^4 + Y^4 + Z^4 + 3a(Z^2Y^2 + Z^2X^2 + Y^2X^2)$$

the first one obtains S_3 the group with the restrictions appearing above in the argument.
The second curve is the Klein quartic, one can obtain that the automorphism group is $PSL_2(\mathbb{F}_7)$.
Easily has as a subgroup of automorphism $C_4^2 \rtimes S_3$ of order 96, therefore can not be bigger for Hurwicz bound.
The fourth one, if $a = 0$ is the Fermat's curve, or $a = \frac{1}{2}(-1 \pm \sqrt{-7})$ is isomorphic to Klein curve. If a does not take this values, easily a subgroup of the $Aut(C)$ is sign of changes of the variables and permutations of variables, this is a group of order 24 isomorphic to S_4. To obtain that this is the full group of automorphism, we need a more careful study on Weierstrass points and the automorphism in

$PGL_3(K)$.

Case 3: G is a simple group.
There are only two transitive primitive groups of PGL_3, one is $PGL_3(\mathbb{F}_2)$ given the Klein quartic (see next talk), already considered.
The other has order bigger than 168, therefore can not be $Aut(C)$ of any genus 3 curve.

$\square$

2.2. The determination of $Aut(C)$ by cyclic covers

Suppose that C is a non-hyperelliptic non-singular projective genus 3 curve, and suppose that C has an automorphism σ. C/σ has genus 0, or 1, (2 can not appear).

If $Aut(C)$ has an element of order ≥ 4 then $genus(C/\sigma) = 0$.

Two situations:

1. C curves which are a Galois cyclic cover of a projective line.

2. C curves which are a Galois cyclic cover of an elliptic curve but not of a projective line.

Let m denote the order of a cyclic group.

1. **cyclic covers over a projective line**.
C Galois cyclic cover of order then $K(C) = K(x, y)$ with $y^m \in K(x)$, therefore:

$$y^m = (x - a_1)^{n_1} \cdot \ldots \cdot (x - a_r)^{n_r}$$

with $1 \leq n_i < m$ and $\sum_{i=1}^{r} n_i$ is divided by m $a_1, \ldots, a_r$ are the points over which the ramification occurs.
Apply now proposition Hurwicz (1893) with $g = 3$ and $\tilde{g} = 0$, we now already that

$$m \leq 20$$

Theorem 16 (Kubayashi-Komiya). *The genus 3 curves C which are projective and non-singular which are also a Galois cyclic cover of order m (can have also a cyclic cover of order a multiple of m) of a projective line and with C non-hyperelliptic are listed as follow (with the equation model up to isomorphism):*

m	$Equation$
3	$y^3 = x(x-1)(x-\alpha)(x-\beta)$
4	$y^4 = x(x-1)(x-\alpha)$
6	$y^3 = x(x-1)(x-\alpha)(x-(1-\alpha))$
7	$y^3 + yx^3 + x = 0$
8	$y^4 = x(x^2-1)$
9	$y^3 = x(x^3-1)$
12	$y^4 = x^3 - 1$

,

Observe that each equation above in $\mathbb{P}^2$ becomes a non-singular quartic.

Make a concrete situation, how proofs goes:

We now that $m \leq 20$. From Hurwicz formula from the cover $C \to C/C_m$ we can not consider $m = 5, 11, 13, 17, 19$.

From the conditions of the equation, from the ramification r and the conditions on n_i we can discard $m = 15, 16, 18, 20$.

Take $m = 8$ (for example here).
The values of v_i can be only divisors of 8, then 2,4,8, therefore all the possibilities are

	v_1	v_2	v_3	v_4	v_5
(i)	2	2	2	2	2
(ii)	2	2	4	4	
(iii)	4	8	8		

Case $(i), (ii)$ reducible equation.
Case (iii) three situations:
(1) $y^8 = (x - a_1)^2(x - a_2)^3(x - a_3)^3$
(2) $y^8 = (x - a_1)(x - a_2)(x - a_3)^6$
(3) $y^8 = (x - a_1)^2(x - a_2)(x - a_3)^5$

by a birational transformation $x = X$ and $y = (X - a_1)^{-2}(X - a_2)^{-1}(X - a_3)^{-1}Y$, (2) is birational equivalent to (1) and (2) is an hyperelliptic curve.

Let us normalize the equation (3) as $y^8 = x^2(x - 1)$. One computes a basis of differentials of the first kind $w_1 = y^{-3}dx$, $w_2 = y^{-6}xdx$, $w_3 = y^{-7}xdx$, and writing $x = -X^{-1}Y^4$, $y = Y$ one obtains a canonical model:

$$X^3Z + XZ^3 + Y^4 = 0$$

(and one observes that this quartic is isomorphic to Fermat's quartic $X^4 + Y^4 + Z^4 = 0$).

How obtain from theorem 16 above the full automorphism group?

We use the equations in the projective model and case by case we study the group of elements of $PGL_3(K)$ that fix the quartic.

Basically is study two situations(Klein quartic knows):
1)one with the affine model:$y^3 = x(x-1)(x-t)(x-s)$
2)second with the affine model:$y^4 = x(x-1)(x-t)$.

Let us mention briefly 2)

Theorem 17 (Kuribayashi-Komiya). *The non-hyperelliptic genus 3 curves non-smooth and projective which are a Galois cyclic cover of a projective line of order m are isomorphic to one of the following equations and has the automorphism group associated to it:*

$Equation = \{F(X,Y,Z) = 0\}$	$Aut(C = V$
$Y^3Z + XZ^3 + X^3Y = 0$	$PGL_2(\mathbb{$
$Y^4 - X^3Z - XZ^3 = 0$	$(C_4 \times C_4)$
$Y^3Z - X^4 + XZ^3 = 0$	C_9
$Y^4 - X^3Z + Z^4 = 0$	$C_4 \circledcirc A$
$Y^4 - X^3Z + (\alpha - 1)X^2Z^2 - \alpha XZ^3 = 0$	$C_4 \circledcirc (C_2 \times$
$Y^3Z - X(X - Z)(X - \alpha Z)(X - (1 - \alpha)Z) = 0$	C_6
$Y^3Z - X(X - Z)(X - \alpha Z)(X - \beta Z) = 0$	C_3

2. **Cyclic cover of a torus.**
   We remember that the automorphism group has a cyclic element σ of order $m > 4$ then the genus of $C/ < \sigma >$ is zero and therefore a cyclic cover of a projective line, as we are done.

   Let us impose that $m = 2, 3$ or 4.
   $n = |Aut(C)|$
   Impose that $n > 4$ in this talk.
   Because $n > 4$ $C/Aut(C) = \mathbb{P}^1$ (Hurwicz).

(Hurwicz) Galois cover $\pi : C \to C/Aut(C)$ verifies:

(a) If $r \geq 5$, then $n \leq 8$ and:
(1) $n = 8$, $v_1 = v_2 = v_3 = v_4 = v_5 = 2$;
(2) $n = 6$, $v_1 = v_2 = v_3 = v_4 = 2$, $v_5 = 3$.

(b) If $r = 4$ then $n \leq 24$ and:
(1) $n = 24$, $v_1 = v_2 = v_3 = 2$, $v_4 = 3$
(2) $n = 16$, $v_1 = v_2 = v_3 = 2$, $v_4 = 4$
(3) $n = 12$, $v_1 = v_2 = 2$, $v_3 = v_4 = 3$
(4) $n = 8$, $v_1 = v_2 = 2$, $v_3 = v_4 = 4$
(5) $n = 6$, $v_1 = v_2 = v_3 = v_4 = 3$.

(c) If $r = 3$, then $n \leq 48$ and:
(1) $n = 48$, $v_1 = v_2 = 3$,$v_3 = 4$
(2) $n = 24$, $v_1 = 3$, $v_2 = v_3 = 4$
(3) $n = 16$, $v_1 = v_2 = v_3 = 4$.

We need a study case by case of every situation. To show the ideas that appears in this study let us take the situation with $r \geq 5$ and $n = 6$.

$n = 6$ ramification 2,2,2,2,3 non-hyperelliptic.

We have an involution σ (bielliptic)

P_1 and P_2 branch points with multiplicity 3

τ the automorphism of order 3 by which P_1 and P_2 are fixed.

$\tau\sigma = \sigma\tau^2$ (cyclic already studied) and $gen(C/ < \tau >) = 1$ is an elliptic curve (=0,studied,=2(No,Hurwicz)).

We use some facts on divisors.

Lemma 18. *Let C be a projective non-singular curve of genus g (≥ 3) and let ι an automorphism of C such that $C/<\iota>$ is an elliptic curve. Denote by v_P the ramification multiplicity of a branch point of the covering $\pi : C \to C/<\iota>$. Then the divisor $\sum(v_P - 1)P$ is canonical.*

This is not useful for our concrete situation $n = 6$ but yes in others and let here to write it.

Lemma 19. *Let C be a non-hyperelliptic genus 3 curve, projective and non-singular. Assume that C has an automorphism ι of order 4 and ι has fixed points on C. Then the $v(\iota) = 4$, denote by P_1, P_2, P_3 and P_4 this four fixed points. Moreover we have that $\sum_{i=1}^{4} P_i$ and $4P_i$ $1 \leq i \leq 4$ are canonical divisors.*

Go back to $n = 6$.
$2(P_1 + P_2)$ is canonical divisor.
$G = \{1, \tau, \tau^2, \sigma = \sigma_1, \sigma_2 = \tau\sigma_1, \sigma_3 = \tau^2\sigma_1\}$, where σ_i are involutions (all bielliptic).

$\{Q_i^{(1)}\}$, $\{Q_i^{(2)}\}$, $\{Q_i^{(3)}\}$ set of 4 fixed points by $\sigma_1, \sigma_2, \sigma_3$ respectively.
$\sum_{i=1}^4 \{Q_i^{(1)}\}$, $\sum_{i=1}^4 \{Q_i^{(2)}\}$ and $\sum_{i=1}^4 \{Q_i^{(3)}\}$ are canonical divisors.

From $\sigma_1\sigma_2\sigma_1 = \sigma_3$:

$$\sigma_1(\sum_{i=1}^4 \{Q_i^{(2)}\}) = \sum_{i=1}^4 \{Q_i^{(3)}\}$$

and $\sigma_1 P_1 = P_2$.
Define meromorphic functions:

$$div(x) = \sum_{i=1}^4 \{Q_i^{(2)}\} - 2(P_1 + P_2)$$

$$div(y) = \sum_{i=1}^4 \{Q_i^{(3)}\} - 2(P_1 + P_2)$$

they verify $\sigma_1(x) = \alpha y$ and $\sigma_1(y) = \beta x$, $\alpha\beta = 1$.(involution)

Rewrite y instead of αy.
Check that $1, x, y$ are a basis for $L(2P_1 + 2P_2)$ with $\tau(x) = -y$ and $\tau(y) = x - y$.
Make change

$$x_1 = \frac{x - 2y + 1}{x + y + 1},\ y_1 = \frac{-2x + y + 1}{x + y + 1},$$

the action of σ_1 and τ is:

$$\sigma_1 : (x_1, y_1) \mapsto (y_1, x_1),$$

$$\tau : (x_1, y_1) \mapsto (y_1/x_1, 1/x_1),$$

$1, x_1, y_1$ basis for $L(K)$ (K canonical divisor) with homogenous coordinates the group acts by

$$\sigma_1 \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\tau \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

then the equation is invariant for the group S_3

therefore the equation are

$$A(X^4+Y^4+Z^4)+B(X^3Y+Y^3X+Z^3X+X^3Z+$$

$$Z^3Y+Y^3Z)+C(X^2Y^2+Y^2Z^2+X^2Z^2)=0$$

for some A, B, C.

If $B=C=0$ and $A\neq 0$ is isomorphic to $y^4=x(x^2-1)$ which has cyclic cover of a projective line, this is already studied.

If $B=0$ and $AC\neq 0$ has a group of order 24, except when $C/A=3\mu$ with $\mu\in\{\frac{-1\pm\sqrt{-7}}{2}\}$ where for this concrete situation is isomorphic to the Klein quartic (studied above).

$ABC\neq 0$ obtain that full group of automorphism are G.

Working situation by situation Kuribayashi and Komiya obtain:

Theorem 20 (Kuribayashi-Komiya). *The non-hyperelliptic genus 3 curves non-smooth and projective which are a Galois cyclic cover of an elliptic curve and not of a projective line are isomorphic to one of the following equations and has the automorphism group associated to it:*

$Equation = \{F(X,Y,Z) = 0\}$
$X^4 + Y^4 + Z^4 + 3a(X^2Y^2 + X^2Z^2 + Z^2Y^2) = 0$
$X^4 + Y^4 + aX^2Y^2 + b(X^2Z^2 + Y^2Z^2) + Z^4 = 0$
$(X^4 + Y^4 + Z^4) + c(X^2Y^2 + Y^2Z^2 + X^2Z^2)+$ $b(X^3Y + Y^3X + Z^3X + X^3Z + Z^3Y + Y^3Z) = 0$
$X^4 + Y^4 + Z^4 + 2aX^2Y^2 + 2bX^2Z^2 + 2cY^2Z^2 = 0$
$a(X^4 + Y^4 + Z^4) + b(X^3Y - Y^3X) + cX^2Y^2 + d(X^2Z^2 + Y^2Z^2) = 0$
$a(X^4 + Y^4 + Z^4) + b(X^3Y + Y^3Z + XZ^3) + c(Y^3X + X^3Z + Y^3Z)+$ $d(X^2Y^2 + X^2Z^2 + Y^2Z^2) = 0$
$(X^4 + Y^4 + Z^4) + Y^2(a_0X^2 + a_1XZ + bZ^2)+$ $(a_2X^3Z + a_3X^2Z + a_4XZ^3) = 0$

2.3. Final remarks

C be a curve of genus ≥ 2.
H subgroup of $Aut(C)$
consider the cover $C \to C/H$, $g_0 = genus(C/H)$:

$$2(g-1)/|H| = 2(g_0 - 1) + \sum_{i=1}^{r} (1 - \frac{1}{m_i}),$$

the signature associate to this cover is

$$(g_0; m_1, \ldots, m_r)$$

where we have exactly r ramification points.

We can list all the possible signatures from Hurwicz, and try to obtain of all this list the ones with $H = Aut(C)$. (above use an intermediate step with cyclic subgroup).

One can obtain:

$Aut(C)$	g_0;signature
$PSL_2(\mathbb{F}_7)$	$(0; 2, 3, 7)$
S_3	$(0; 2, 2, 2, 2, 3)$
C_2	(1;2,2,2,2)
$C_2 \times C_2$	$(0; 2, 2, 2, 2, 2, 2)$
Q_8	$(0; 2, 2, 2, 2, 2)$
S_4	(0;2,2,2,3)
$C_4^2 \rtimes S_3$	(0;2,3,8)
$C_4 \circledcirc (C_2)^2$	(0;2,2,2,4)
$C_4 \circledcirc A_4$	(0;2,3,12)
C_3	$(0; 3, 3, 3, 3, 3)$
C_6	(0;2,3,3,6)
C_9	(0;3,9,9)

M_g moduli space of genus g curves.
$M_{g,r}$ moduli space of genus g curves with r distint marked points
Is known that

$$dim(M_{g,r}) = 3g - 3 + r.$$

Is known that the dimension in M_g of the connected components of an appearing signature $(g_0; \alpha_1, \ldots, \alpha_r)$ is $dim(M_{g_0,r})$, therefore:
Remark 21. *There a lot of non-hyperelliptic genus 3 curves that has no automorphism, in particular the generic curve for* M_3 *should has no automorphism.*

C has a large automorphism group if its point in M_g has a neighborhood such that any other curve in this neighborhood has an automorphism group a group with less elements than the group that has the curve C.

Corollary 22. *Let C be a curve defined over $\mathbb{C}$ ($g \geq 2$). Then: C has a large automorphism group if and only if exists a Bely$\hat{\imath}$ function defining a normal covering $\pi : C \to \mathbb{P}^1$.*

If we center now in our tables for genus 3 curves:

C non-hyperelliptic genus 3 curves with large automorphism group:

C	$Aut(C)$
$Z^3Y + Y^3X + X^3Z$	$PSL_2(\mathbb{F}_7)$
$Z^4 + X^4 + Y^4$	$C_4^2 \rtimes S_3$
$Z^4 + YX^3 + Y^3X$	$C_4 \circledcirc A_4$
$Z^4 + ZY^3 + YX^3$	C_9