Local polynomial factorisation

improving the Montes algorithm

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cnrs

My background

Singularities of plane algebraic curves

Puiseux series, Newton-Puiseux algorithm

Puiseux series

$\mathrm{Char}(\mathbb{K}) = 0$ or $\mathrm{Char}(\mathbb{K}) > d := deg(F)$

Theorem (Puiseux, 1850)

$F \in \mathbb{K}[[t]][x]$ has d distinct roots in $\overline{\mathbb{K}((t))}$:

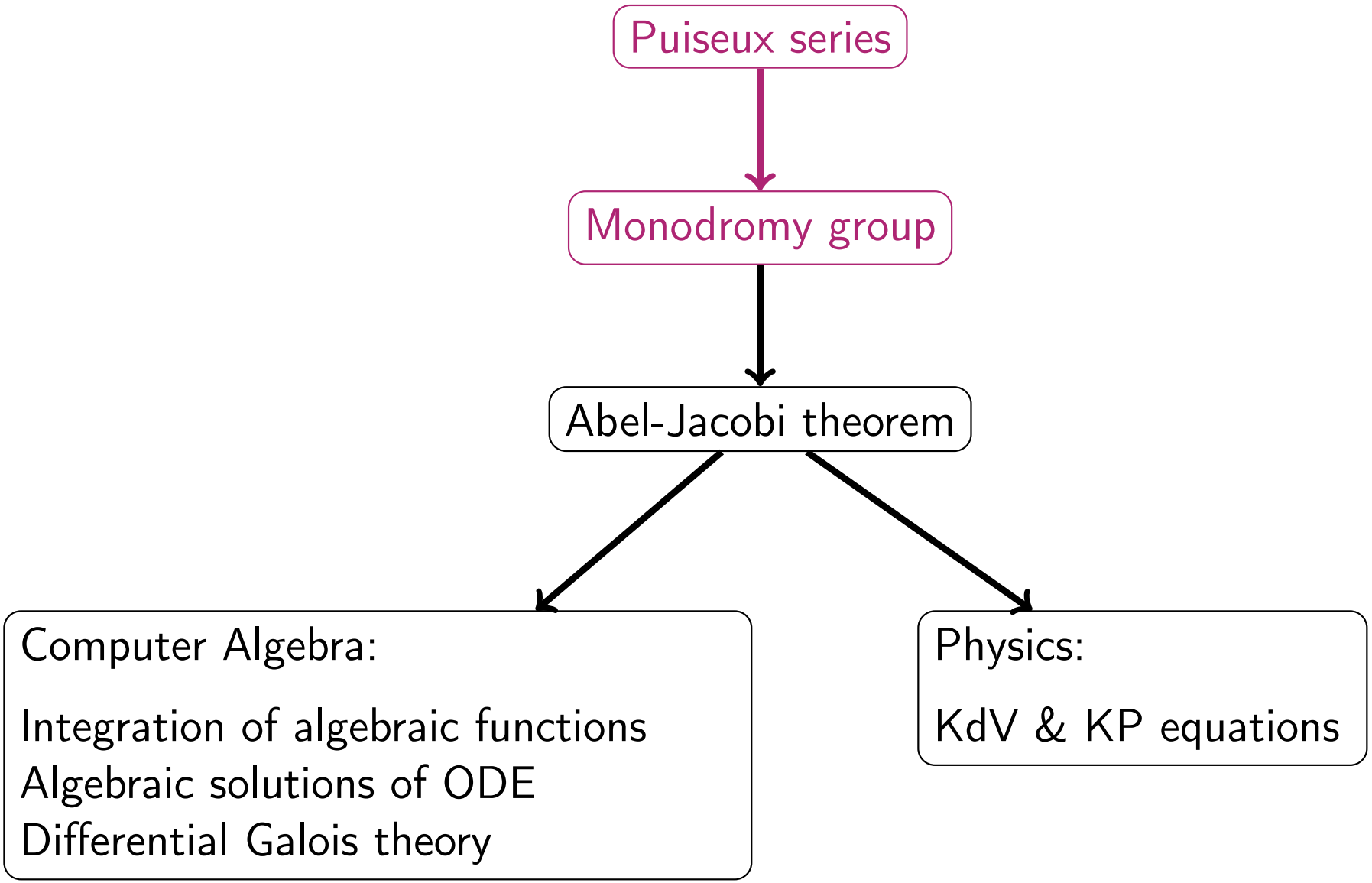
$$S_{ij}(t) = \sum_{k=n_i}^{\infty} \alpha_{i,k}\, \zeta_{e_i}^{jk}\, t^{\frac{k}{e_i}}$$

- $e_1, \cdots, e_s \in \mathbb{N}^\star$ *and* $d = \sum_{i=1}^{s} e_i$
- $0 \leq j \leq e_i - 1$, $1 \leq i \leq s$, $n_i \in \mathbb{Z}$, $\alpha_{i,n_i} \neq 0$
- ζ_{e_i} *primitive e_i-th root of unity.*

Moreover, $\{\alpha_{i,k}\}$ belongs to a finite extension of $\mathbb{K}$.

$$\overline{\mathbb{K}((x))} = \bigcup_{e \in \mathbb{N}^\star} \overline{\mathbb{K}}((x^{\frac{1}{e}}))$$

My initial motivations



Computing Puiseux series: tools and idea

$$F(t,x) = x^6 + t\,x^5 + 5\,t^3\,x^4 - 2\,t\,x^4 + 4\,t^2\,x^2 + t^5 - 3\,t^4$$

$\Longrightarrow$ We're looking for $S(t) = \alpha\,t^{\frac{m}{q}} + \cdots$ s.t. $F(t, S(t)) = 0$

$$\begin{aligned} F(t, \alpha\,t^{\frac{m}{q}} + \cdots) \quad &= \alpha^6\,t^{\frac{6\,m}{q}} + \alpha^5\,t^{\frac{5\,m}{q}+1} + 5\,\alpha^4\,t^{\frac{4\,m}{q}+3} \\ &\quad -2\,\alpha^4\,t^{\frac{4\,m}{q}+1} + 4\,\alpha^2\,t^{\frac{2\,m}{q}+2} + t^5 - 3\,t^4 + \cdots \end{aligned}$$

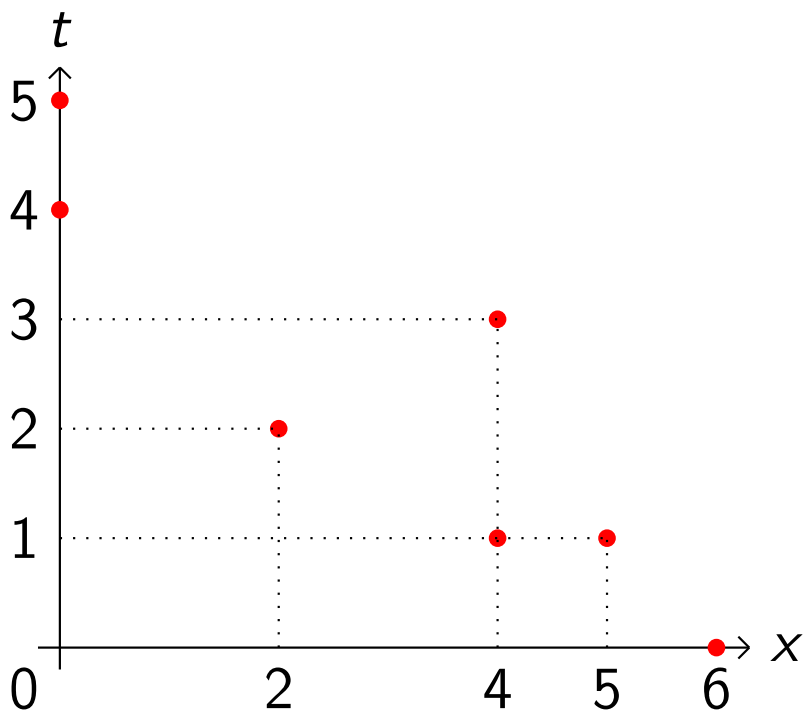
- Some of these terms must cancel !

$\Longrightarrow$ (m, q) s.t. at least two exponents are the same

Support of the polynomial

$$F(t,x) = t^0\,x^6 + t^1\,x^5 + 5\,t^3\,x^4 - 2\,t^1 x^4 + 4\,t^2\,x^2 + t^5\,x^0 - 3\,t^4\,x^0$$

- Supp(F)$= \{(i,j) \in \mathbb{N}^2 \mid a_{ij} \neq 0\}$



Choice of (m, q) that increases the t-order ?

$$F(t,x) = x^6 + t\,x^5 + 5\,t^3\,x^4 - 2\,t\,x^4 + 4\,t^2\,x^2 + t^5 - 3\,t^4$$

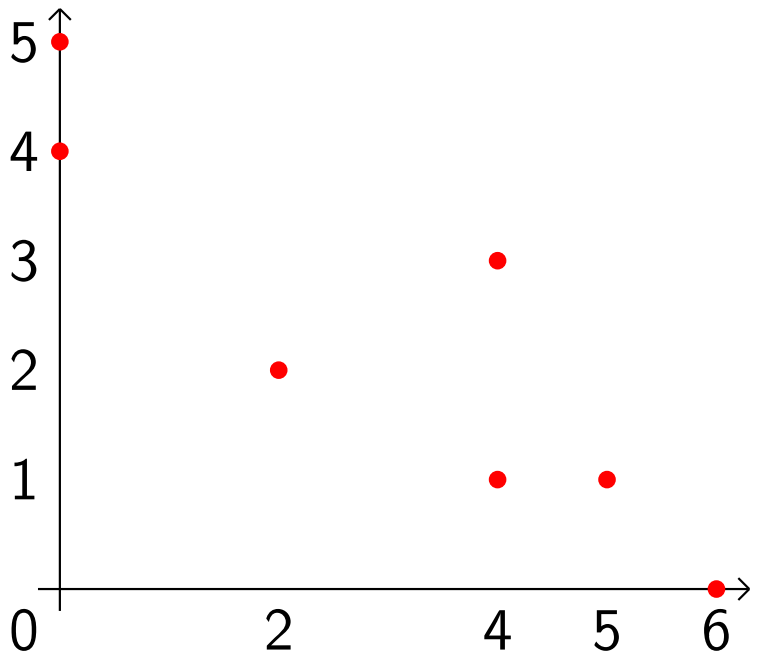
- Supp(F)$= \{(i,j) \in \mathbb{N}^2 \mid a_{ij} \neq 0\}$

  $*$ (m, q) for cancelling two terms ?

$\rightsquigarrow$ at least two points on $m\,i + q\,j = l$

  $*$ increasing the t-order ?

$\rightsquigarrow$ no other point under the line



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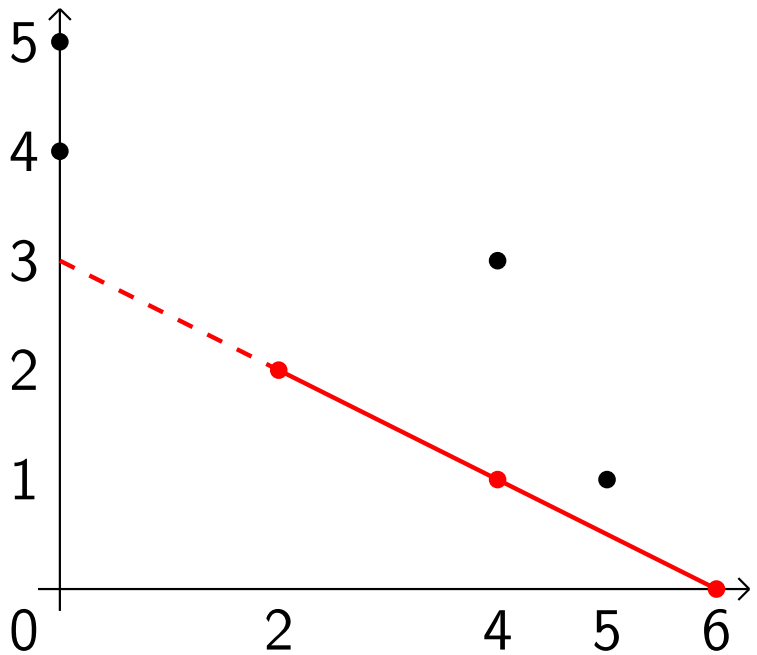
(Δ_1) $i + 2\,j = 6$ is such a line

$$\rightsquigarrow S(t) = \alpha\,t^{1/2} + \cdots$$



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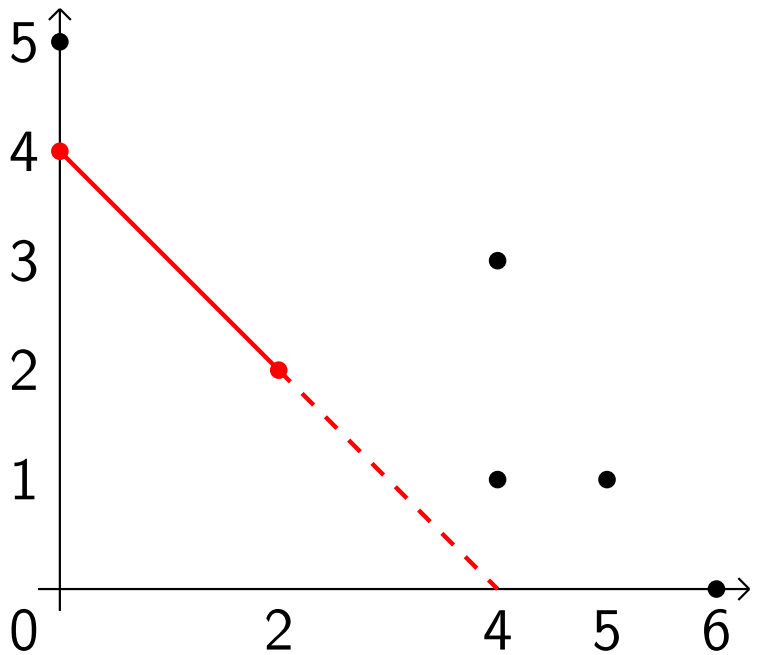
(Δ_1) $i + 2\,j = 6$ is such a line

(Δ_2) $i + j = 4$ is too



Newton polygon

$$F(t,x) = x^6 + t\,x^5 + 5\,t^3\,x^4 - 2\,t\,x^4 + 4\,t^2\,x^2 + t^5 - 3\,t^4$$

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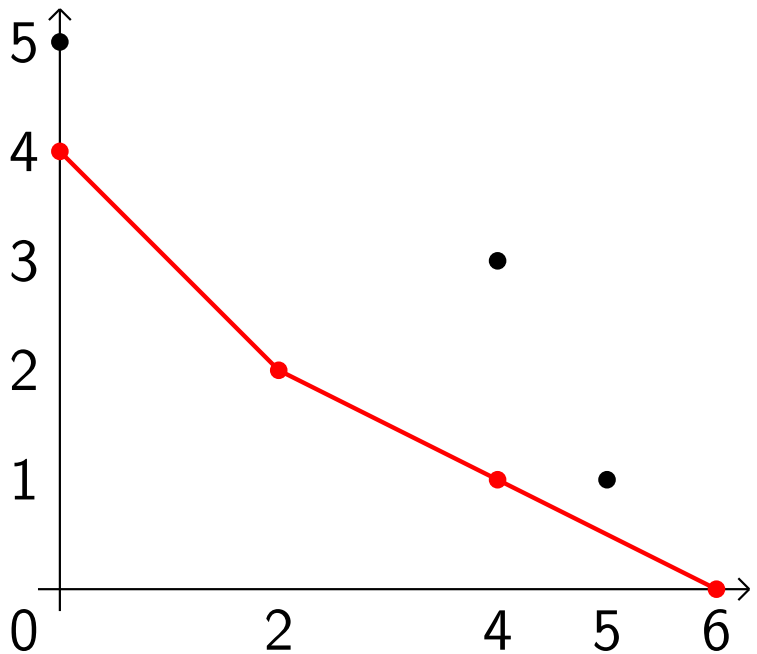
— $\mathcal{N}(F)$: lower part of convex hull of Supp(F).



Choice of α that increases the t-order ?

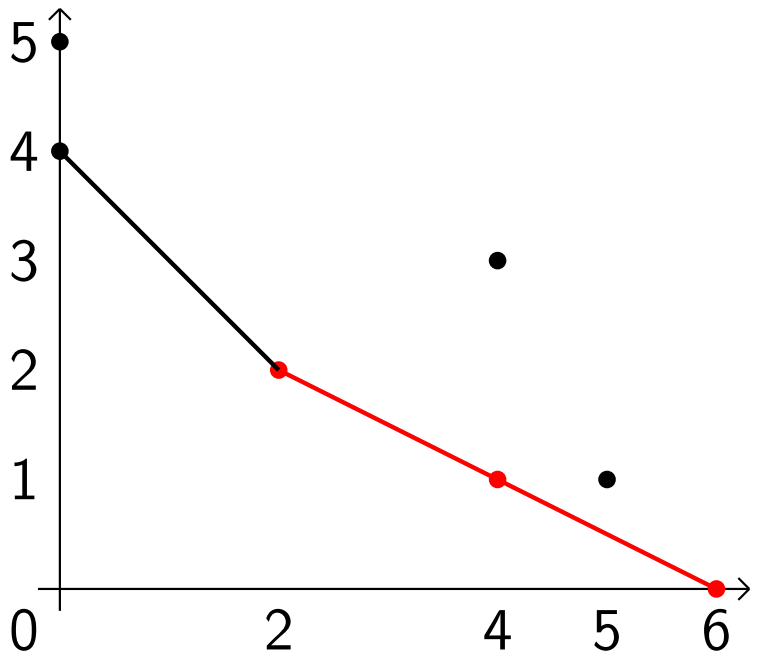
$$F(t,x) = x^6 + t\,x^5 + 5\,t^3\,x^4 - 2\,t\,x^4 + 4\,t^2\,x^2 + t^5 - 3\,t^4$$

- Supp(F)$= \{(i,j) \in \mathbb{N}^2 \mid a_{ij} \neq 0\}$

— $\mathcal{N}(F)$: lower part of convex hull of Supp(F).

$$F(T^2, \alpha\,T) = (\alpha^6 - 2\alpha^4 + 4\alpha^2)\,T^6$$

$$+\alpha^5\,T^7 - 3\,T^8 + (5\alpha^4 + 1)\,T^{10} + \ldots$$



Characteristic polynomial

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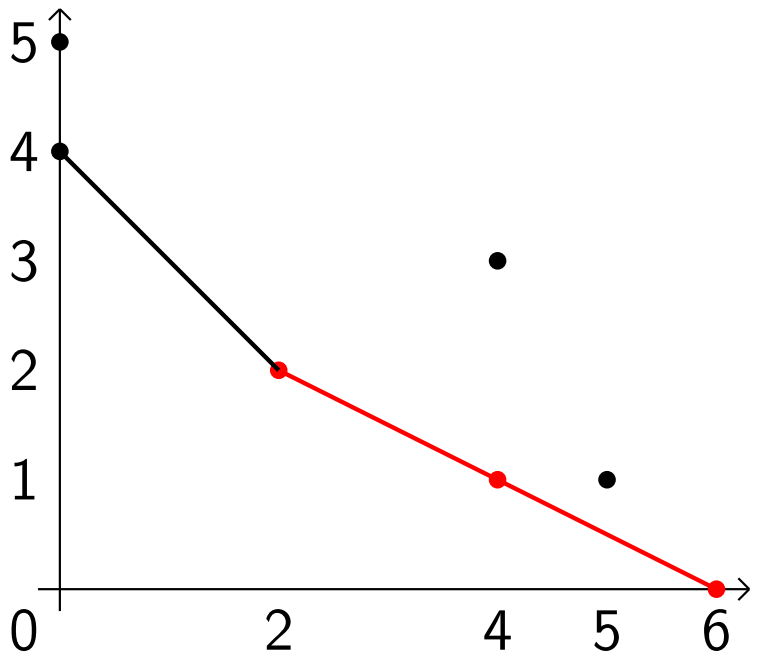
Characteristic polynomial:

$$\phi_{\Delta_1}(\beta) = \beta^2 - 2\,\beta + 4$$



Newton-Puiseux algorithm

For each edge Δ of $\mathcal{N}(F)$

1. Factor $\phi_\Delta = \prod_{k=1}^{s} \phi_k^{M_k}$

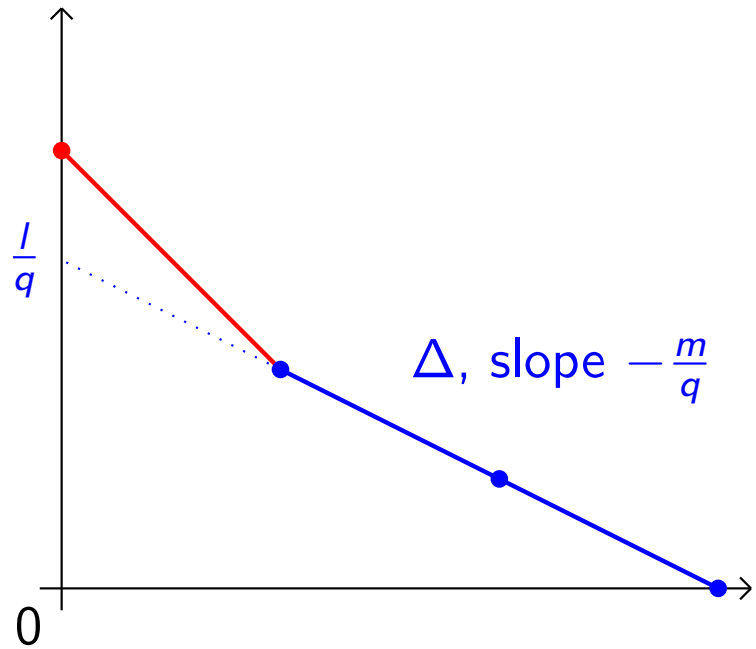
2. For each ϕ_k,

   Puiseux transformation:

   $$F_{\Delta,\xi}(t,x) = \frac{F(t^q, t^m(x+\xi^{\frac{1}{q}}))}{t^l}$$

   with ξ root of ϕ_k.

3. Recursive calls: $\{F_{\Delta,\xi}(t,x)\}_{\Delta,\xi}$



Main results

- [Po08] A modular-numeric strategy.
  1. Compute S modulo a *well chosen* prime $p \rightsquigarrow$ essential terms.
  2. Use this computation to get numerical coefficients of the series.
- Improved arithmetic complexity.
  - [Po08] $\mathcal{O}(D^8) \rightsquigarrow \mathcal{O}^{\sim}(D^5)$.
  - [PoRy15] $\mathcal{O}^{\sim}(D^4)$: *Abhyankhar's trick* (d-th approximate root).
  - [PoWe21] $\mathcal{O}^{\sim}(D^3)$: divide and conquer strategy.

Application: genus in an expected $\mathcal{O}^{\sim}(D^3)$ *binary* operations.

Moving towards generalised Newton polygons

improving the irreducibility test

One example

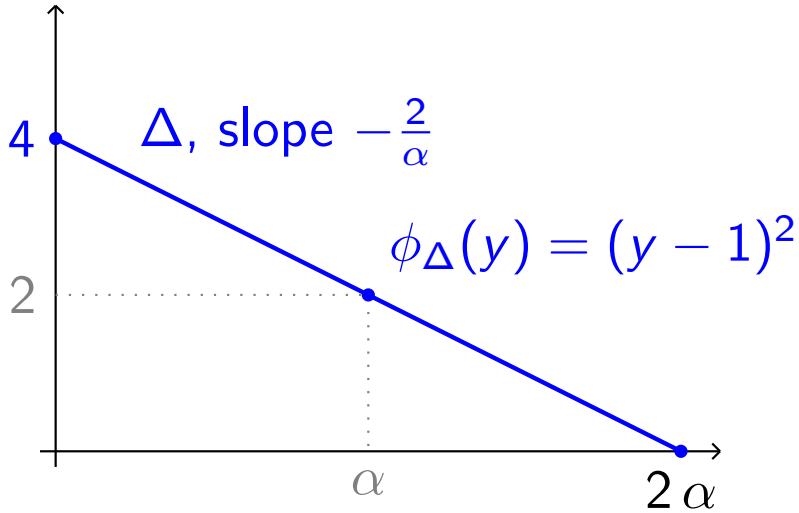
$$F = (x^\alpha - t^2)^2 + t^\alpha \in \mathbb{A}[x] \text{ with } \mathbb{A} = \mathbb{C}[[t]]$$

- $d = \deg(F) = 2\,\alpha$,
- $\delta = v_t(\mathrm{Disc}(F)) = 2\,\alpha^2 - 4\,\alpha + 4$.
- Assume $\alpha > 4$ odd.

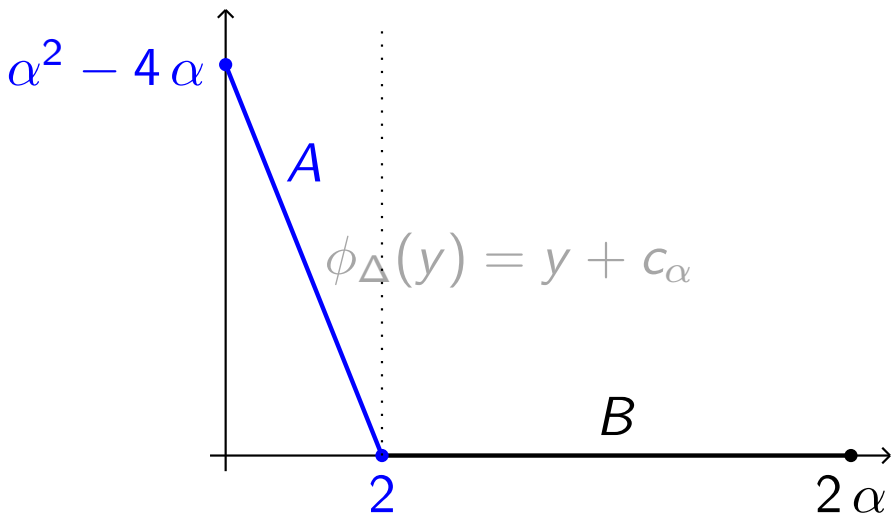
Is F irreducible in $\mathbb{C}[[t]][x]$?

Using the Newton-Puiseux algorithm. [PoWe21]

$$F = y^{2\alpha} - 2x^2 y^\alpha + x^4 + x^\alpha$$



1. $G \leftarrow F(t^\alpha, t^2(x+1))/t^{4\alpha}$,
2. Hensel: $G = A \cdot B$,
3. Recursive call with A

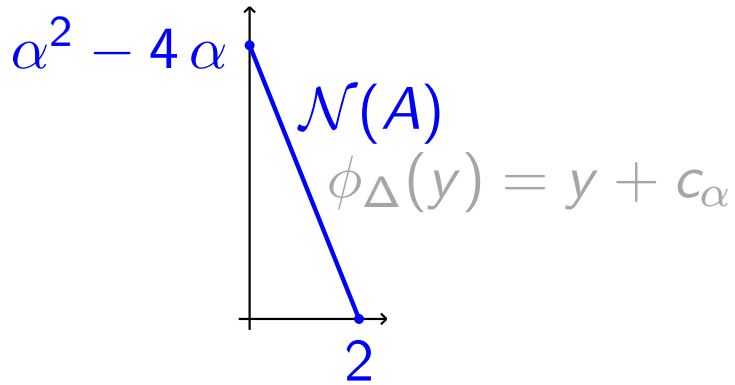


Polynomial	size
F	$\Theta(\alpha^2) = \Theta(\delta)$
G	$\Theta(\alpha^3) = \Theta(d\,\delta)$
A	$\Theta(\alpha^2) = \Theta(\delta)$

Answer: Yes complexity: $\Theta(d\,\delta)$ Answer in $\mathcal{O}(\delta)$?

Avoiding any blow-up ?

$$F = (x^\alpha - t^2)^2 + t^\alpha$$



- Writing $\psi = x^\alpha - t^2$, we have $F = \psi^2 + t^\alpha$,
  - Can we "guess" the second Newton polygon from $\psi^2 + t^\alpha$?
  - Can we "read" ϕ_Δ ?

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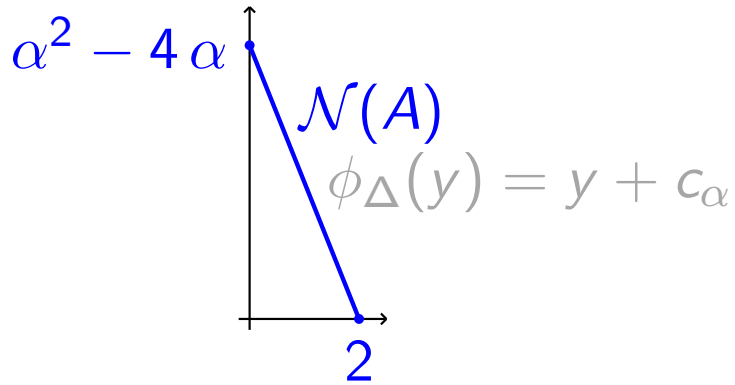

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- Key ingredients:
  - $\psi = \sqrt[2]{F}$ is an approximate root of F,
  - $F = \psi^2 + t^\alpha$ is the ψ-adic expansion of F.

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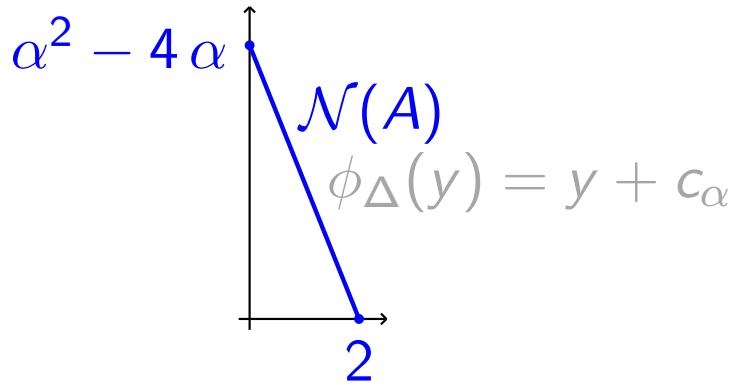

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- Key ingredients:
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- Questions:
  - Why t^α corresponds to $\alpha^2 - 4\,\alpha$?
  - How to recover the correct characteristic polynomial ?

This talk

Context:

- $\mathbb{A}$ a discrete valuation ring (e.g. $\mathbb{K}[[t]]$, $\mathbb{Z}_p$),
- $v_{\mathbb{A}}$ valuation over $\mathbb{A}$ (e.g. v_t, v_p),
- π an uniformiser (e.g. t, p),
- $\mathbb{F}$ the residue field (e.g. $\mathbb{K}$, $\mathbb{F}_p$).

Objective(s):

1. Irreducibility test in $\mathbb{A}[x]$,
2. Factorisation in $\mathbb{A}[x]$.
3. Case $\mathbb{A} = \mathbb{K}[[t]]$: Puiseux series of F ?

Notations: $F \in \mathbb{A}[x]$ (monic) ; $d = \deg(F)$; $\delta = v_{\mathbb{A}}(\mathrm{Disc}(F))$

Approximate root of $F \in \mathbb{A}[x]$ monic [Ab10]

- Hyp: $\mathrm{char}(\mathbb{A})$ does not divide d,
- Let $N \in \mathbb{N}$ dividing d,

Proposition

There is an unique monic $\psi \in \mathbb{A}[x]$ such that:

- $\deg(\psi) = d/N$,
- $\deg(F - \psi^N) < d - d/N$,

$\rightsquigarrow$ $\psi = \sqrt[N]{F}$ is the N-th approximate root of F.

$$F = \sum_{i=0}^{N} a_i \psi^i \ , \ \deg(a_i) < \deg(\psi) \Longrightarrow a_{N-1} = 0$$

Example: $\psi = \sqrt[d]{F} = x + \dfrac{a_{d-1}}{d}$ is the d-th approximate root of F.

Valuations on $\mathbb{A}[x]$

- Gauss valuation:
  - $F = \sum_i a_i\, x^i$,
  - $\upsilon_0(F) = \min_i \upsilon_{\mathbb{A}}(a_i)$.
- Extended valuation: given $\psi \in \mathbb{A}[x]$ monic, $\frac{m}{q} \in \mathbb{Q}$:
  - $\upsilon_\psi = (\upsilon_0, \psi, \frac{m}{q})$ *extends* υ_0.

    Defined by $\upsilon_\psi(\psi) = m\,q$, $\upsilon_\psi(x) = m$ and $\upsilon_\psi(\pi) = q$,
  - Expand $F = \sum_i a_i(x)\,\psi^i$ with $\deg(a_i) < \deg(\psi)$,
  - Generalised Newton polygon:

    $\mathcal{N}_\psi(F)$ is the lower convex hull of $(i, \upsilon_\psi(a_i\,\psi^i) - \upsilon_\psi(F))_i$.

Improving the irreducibility test

generalisation of the work of Abhyankhar to $\mathbb{A}[x]$.

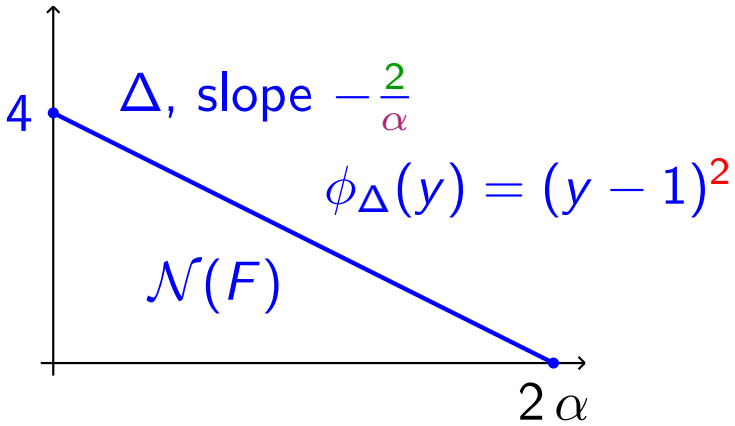
link with the Newton–Puiseux algorithm for $\mathbb{A} = \mathbb{K}[[t]]$

PoWe - Submitted, *A quasi-linear irreduccibility test in* $\mathbb{K}[[x]][y]$

PoWe - Preprint, *Local polynomial factorisation: improving the Montes algorithm*

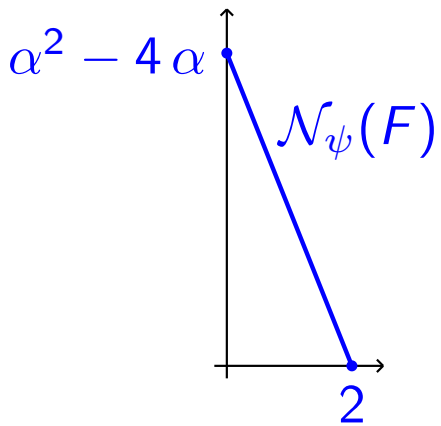
We get the second Newton polygon !

$$F = (x^\alpha - t^2)^2 + t^\alpha$$



With $m = 2$, $q = \alpha$, $\psi = \sqrt[2]{F}$, we get:

- $F = \psi^2 + t^\alpha$.
- $v_\psi(F) = 4\,\alpha$
- $v_\psi(\psi^2) - v_\psi(F) = 0$,
- $v_\psi(t^\alpha) - v_\psi(F) = \alpha^2 - 4\,\alpha$.



Reminder: $v_\psi(t) = \alpha \quad v_\psi(x) = 2 \quad v_\psi(\psi) = 2\,\alpha$

Complexity ?

- Computing $\psi = \sqrt[N]{F}$: $\mathcal{O}(\mathsf{M}(d)) = \mathcal{O}^{\sim}(d)$ op in $\mathbb{A}$.
  - $F_\infty = x^d\, F(1/x)$ the reciprocal polynomial of F,
  - $F_\infty(0) = 1 \rightsquigarrow \exists!\ \phi \in \mathbb{A}[[x]]$ s.t. $\phi(0) = 1$ and $\phi^N = F_\infty$,
  - ϕ is the root of $z^N - F_\infty = 0 \rightsquigarrow$ Newton iteration !
  - ψ is the reciprocal polynomial of $\lceil \phi \rceil^{\frac{d}{N}}$

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- ψ-adic expansion: $\mathcal{O}(\mathsf{M}(d)\log(N)) = \mathcal{O}^{\sim}(d)$ op in $\mathbb{A}$.
  - $F = A\,\psi^{\frac{N}{2}} + B \rightsquigarrow \mathcal{O}(\mathsf{M}(d))$
  - Recursive call on A and B.

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  - Recursive call on A and B.
- Truncation: $n = 2\,\delta/d$.

Total cost: $\delta\,\mathrm{plog}(d)$!

Types and operators.

- $\mathbf{t}_0 = [P_0]$, with $P_0 \in \mathbb{F}[y]$ irreducible

  $\rightsquigarrow v_0$ the Gauss valuation, $R_0(G) = \overline{G(y)/\pi^{v_0(G)}}$, $\mathbb{F}_0 = \mathbb{F}$.

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  - $\mathbf{t}_{k-1}$ is a type.
  - $\phi_k \in \mathbb{A}[x]$ is monic, irreducible and satisfies $R_{k-1}(\phi_k) \sim P_{k-1}$.
  - $\lambda_k = -m_k/q_k \in \mathbb{Q}^-$, with $(q_k, m_k) \in \mathbb{N}^2$ coprime.
  - $P_k \in \mathbb{F}_k[y]$ is monic, irreducible over $\mathbb{F}_k := \mathbb{F}_{k-1}[y]/(P_{k-1})$.

$$\ell_k := \deg(P_k); z_k := y \mod P_k(y) \in \mathbb{F}_{k+1}$$

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  - $P_k \in \mathbb{F}_k[y]$ is monic, irreducible over $\mathbb{F}_k := \mathbb{F}_{k-1}[y]/(P_{k-1})$.
- $v_k(G) := \min_i \left(q_{k-1} v_{k-1}(a_i' \phi_{k-1}^i) + m_{k-1} i\right)$
- $R_k(G) = \sum_{i \in \Delta_k} z_{k-1}^{\tau_{k,i}} R_{k-1}(a_i \phi_k^i)(z_{k-1})\, y^{\frac{i - i_k(G)}{q_k}} \in \mathbb{F}_k[y]$

$$\ell_k := \deg(P_k); z_k := y \mod P_k(y) \in \mathbb{F}_{k+1}$$

Types and operators.

- $\mathbf{t}_0 = [P_0]$, with $P_0 \in \mathbb{F}[y]$ irreducible

  $\rightsquigarrow v_0$ the Gauss valuation, $R_0(G) = \overline{G(y)/\pi^{v_0(G)}}$, $\mathbb{F}_0 = \mathbb{F}$.
- $\mathbf{t}_k = [P_0, (\phi_1, \lambda_1, P_1), \cdots, (\phi_k, \lambda_k, P_k)]$ is a type if:
  - $\mathbf{t}_{k-1}$ is a type.
  - $\phi_k \in \mathbb{A}[x]$ is monic, irreducible and satisfies $R_{k-1}(\phi_k) \sim P_{k-1}$.
  - $\lambda_k = -m_k/q_k \in \mathbb{Q}^-$, with $(q_k, m_k) \in \mathbb{N}^2$ coprime.
  - $P_k \in \mathbb{F}_k[y]$ is monic, irreducible over $\mathbb{F}_k := \mathbb{F}_{k-1}[y]/(P_{k-1})$.
- $v_k(G) := \min_i \left(q_{k-1} v_{k-1}(a_i' \phi_{k-1}^i) + m_{k-1} i\right)$
- $R_k(G) = \sum_{i \in \Delta_k} z_{k-1}^{\tau_{k,i}} R_{k-1}(a_i \phi_k^i)(z_{k-1})\, y^{\frac{i - i_k(G)}{q_k}} \in \mathbb{F}_k[y]$

$$\ell_k := \deg(P_k); z_k := y \mod P_k(y) \in \mathbb{F}_{k+1}$$

ϕ_k approximate root $\Longrightarrow \ell_k\, q_k > 1$ (no refinment step !)

General algorithm

Algorithm `FastIrreducible`(F)**:**

Input: $F \in \mathbb{A}[x]$ monic separable s.t. $\mathrm{Char}(\mathbb{F}) \nmid \deg(F)$.
Output: F irreducible ?, a type **t** and a representative ϕ of **t**.

if $R_0(F)$ *is not some* $P_0^{N_0}$ **then return** *False*, [];
$\mathbf{t} \leftarrow [P_0]$, $k \leftarrow 1$, $N \leftarrow N_0$;
while $N > 1$ **do**
  $\phi_k \leftarrow \sqrt[N]{F}$;
  if $\mathcal{N}_k^-(F)$ *is not one sided* **then return** *(False,* **t**, ϕ_k*)*;
  if $R_k(F)$ *is not some* $P_k^{N_k}$ **then return** *(False,* **t**, ϕ_k*)*;
  $\mathbf{t} \leftarrow \mathbf{t} \cup (\phi_k, \lambda_k, P_k)$; // λ_k `the slope of` $\mathcal{N}_k(F)$
  $N \leftarrow N_k$, $k \leftarrow k + 1$;
return *(True,* **t**, F*)*

Factorisation over $\mathbb{A}[x]$

a new divide and conquer strategy

adapt the Hensel–Newton algorithm to extended valuations

Hensel lemma works with extended valuations

- $\phi = \phi_k$ a representative of a type $\mathbf{t}_{k-1}$ (i.e. $R_{k-1}(\phi) \sim P_{k-1}$),
- $v = v_{k+1}$ an augmented valuation (from v_k, ϕ and some λ_k).

Lemma

Assume $B = \phi^b + \cdots$ *and* $v(B) = b\,v(\phi)$. *Then*

- $v(A\%B) \geq v(A)$,
- $v(A /\!\!/ B) \geq v(A) - v(B)$.

Notation:

$$F = \sum_{B=(b_1,\cdots,b_k)} c_B(x)\,\phi_1^{b_1} \cdots \phi_k^{b_k}$$ the multi-adic expansion of F

$\lceil F \rceil^n \rightsquigarrow$ remove from it the terms of valuations $> n$

Adaptation of the Hensel algorithm

In: $v(F - G\,H) \geq v(F) + n$ and $v(S\,G + T\,H - 1) \geq n$.

Algorithm `HenselStep`:

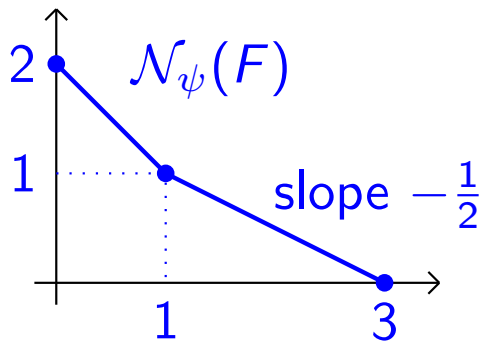
$E \leftarrow \lceil F - G\,H \rceil^{v(F)+2n}$;
$Q, R \leftarrow \lceil \mathtt{QuoRem}(U\,E, H) \rceil^{v(F)+2n}$;
$\tilde{G} \leftarrow \lceil G + E\,V + Q\,G \rceil^{v(G)+2n}$;
$\tilde{H} \leftarrow \lceil H + R \rceil^{v(H)+2n}$;
$B \leftarrow \lceil U\,\tilde{G} + V\,\tilde{H} - 1 \rceil^{2n}$;
$C, D \leftarrow \lceil \mathtt{QuoRem}(U\,B, \tilde{H}) \rceil^{2n}$;
$\tilde{U} \leftarrow \lceil U - D \rceil^{2n-v(G)}$;
$\tilde{V} \leftarrow \lceil V - B\,V - C\,\tilde{G} \rceil^{2n-v(H)}$;
return $\tilde{H}$, $\tilde{G}$, $\tilde{U}$, $\tilde{V}$

Out: $v(F - \tilde{G}\,\tilde{H}) \geq v(F) + 2\,n$ and $v(\tilde{S}\,\tilde{G} + \tilde{T}\,\tilde{H} - 1) \geq 2\,n$.

Good initialisation ?

$$F = \underbrace{\psi^3}_{18} + \underbrace{\pi^3 x^2 \psi}_{19} + \underbrace{\pi^6 x}_{20} \text{ with } \psi = x^3 - \pi^2$$

- $v_\psi(\pi) = 3$, $v_\psi(x) = 2$, $v_\psi(\psi) = 6$.

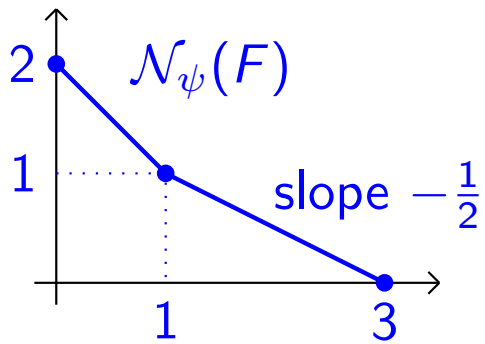


Good initialisation ?

$$F = \underbrace{\psi^3}_{39} + \underbrace{\pi^3 x^2 \psi}_{39} + \underbrace{\pi^6 x}_{40} \text{ with } \psi = x^3 - \pi^2$$

- $v_\psi(\pi) = 3,\ v_\psi(x) = 2,\ v_\psi(\psi) = 6.$
- Extend v_ψ from right hand slope:

$$v(\pi) = 6,\ v(x) = 4,\ v(\psi) = 13$$

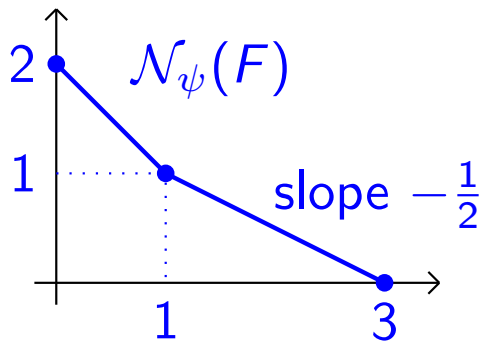


$$v(\pi) = q\, v_\psi(\pi),\ v(x) = q\, v_\psi(x),\ v(\psi) = q\, v_\psi(\psi) + m$$

Good initialisation ?

$$F = \underbrace{\psi^3}_{39} + \underbrace{\pi^3\, x^2\, \psi}_{39} + \underbrace{\pi^6\, x}_{40} \text{ with } \psi = x^3 - \pi^2$$

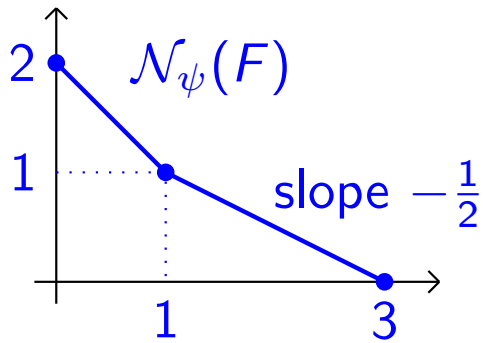
- $v_\psi(\pi) = 3,\ v_\psi(x) = 2,\ v_\psi(\psi) = 6.$
- Extend v_ψ from right hand slope:
$$v(\pi) = 6,\ v(x) = 4,\ v(\psi) = 13$$



- $\tilde{R}(F) = y\,(y^2 + 1)$
- $H_0 = \overbrace{\psi}^{13},\ G_0 = \overbrace{\psi^2 + \pi^3\, x^2}^{26} \Longrightarrow \overbrace{F}^{39} - H_0\, G_0 = \overbrace{\pi^6\, x}^{40}$

Good initialisation ?

$$F = \underbrace{\psi^3}_{39} + \underbrace{\pi^3\, x^2\, \psi}_{39} + \underbrace{\pi^6\, x}_{40} \text{ with } \psi = x^3 - \pi^2$$



- $v_\psi(\pi) = 3,\ v_\psi(x) = 2,\ v_\psi(\psi) = 6.$
- Extend v_ψ from right hand slope:
  $$v(\pi) = 6,\ v(x) = 4,\ v(\psi) = 13$$
- $\tilde{R}(F) = y\,(y^2 + 1)$
- $H_0 = \overbrace{\psi}^{13},\ G_0 = \overbrace{\psi^2 + \pi^3\, x^2}^{26} \Longrightarrow \overbrace{F}^{39} - H_0\, G_0 = \overbrace{\pi^6\, x}^{40}$
- With $s_0 = 1$ and $t_0 = -y$, we have $s_0\,(y^2 + 1) + t_0\, y = 1$,
- $S_0 = \underbrace{\pi^{-5} x}_{-26},\ T_0 = \underbrace{-\pi^{-5} x\, \psi}_{-13} \Longrightarrow S_0\, G_0 + T_0\, H_0 - 1 = \underbrace{\pi^{-2} \psi}_{1}$

Multi-factor Hensel lifting

(modified residual polynomial: $\tilde{R}_k(G)(y) := y^{i_k(G)} R_k(G)(y^{q_k})$)

Main algorithm:

Given
- a factorisation $\tilde{R}(F) = h_0 h_1 \cdots h_r h_\infty$; $h_0 = y^s$, $h_\infty \in \mathbb{F}_k^\times$,
- a precision $n \in \mathbb{N}$.

Compute a factorisation (use a subproduct tree)

$$v(F - F_0 F_1 \cdots F_r F_\infty) > n \quad \text{with} \quad \tilde{R}(F_i) = h_i, \quad i = 0, \ldots, t, \infty.$$

Remark: $F_0, \cdots, F_r$ of type $\mathbf{t}$; $\mathbf{t}$ does not divide F_∞

Complexity: $\mathcal{O}\left(\frac{v(F)+n}{v(\pi)}\, d\right)$

Factorisation with precision σ in $\mathbb{A}[x]$: $\mathcal{O}(\sigma\, d + \delta)$ if $\deg(F_\infty) \leq \frac{d}{2}$.

A factorisation algorithm.

1. Run `FastIrreducible` with precision $2\delta/d$ $\qquad \mathcal{O}(\delta)$

   $\rightsquigarrow$ either F irreducible (return F), either get a type $\mathbf{t}_{k-1}$.

2. Factor $\tilde{R}_k(F) = h_0 h_1 \cdots h_r$ in $\mathbb{F}_k[y]$.

3. Lift $F = F_0 F_1 \cdots F_r$ with precision σ. $\qquad \mathcal{O}(\sigma\, d + \delta)$

4. Go back to Step 1 for each F_i.

Total: $\qquad \mathcal{O}(\rho\,(\sigma\, d + \delta))$.

A divide and conquer algorithm.

1. Find a type $\mathbf{t}$ such that $F_{\mathbf{t}}$ has degree $> d/2$ and that either $F_{\mathbf{t}}$ is irreducible, either any of its proper factor has degree $\leq d/2$.

   Previous algorithm with precision $4\,\delta/d$ $\qquad \mathcal{O}(\rho\,\delta)$

2. Hensel $\rightsquigarrow F = F_0\,F_1 \cdots F_r F_\infty$ $\qquad \mathcal{O}(n\,d + \delta)$

3. Recursive call for each non irreducible factor.

Total: $\qquad \mathcal{O}((n+\delta)\,d)$

State of the art (sketch)

- Abhyankar-Moh [Ab06]: approximate roots,
- Mac Lane, Abhyankar [Ma36^2,Ab90,Ru14]: extended valuations,
- Montes et al [Mo99,GuMoNa11&12,BaNaSt13,GuNaPa12] $\mathcal{O}(d^2 + d\delta^2)$,
- Caruso et al [CaRoVa16]: slope factorisation,

Case $\mathbb{A} = \mathbb{K}[[x]]$:

- Sasaki et al [KaSa99,AlAtMa17]: Extended Hensel Construction at least $\mathcal{O}(d^2(\delta + d^2))$,
- Puiseux [PoRy15,PoWe]: Newton–Puiseux algorithm $\mathcal{O}(d\,\delta)$.

Conclusion

Up to factorisations in $\mathbb{F}_k[y]$:

- Irreducibility test in $\mathbb{A}[x]$ in $\mathcal{O}(\delta)$, $\leftarrow$ improved by a factor d !
- "direct" factorisation in $\mathbb{A}[x]$: $\mathcal{O}(\rho\, n\, d)$, $\leftarrow$ was $\mathcal{O}(n\, d^2)$
- Divide and conquer: $\mathcal{O}((n+\delta)\, d)$.

Application: OM-factorisation for $\mathbb{A} = \mathbb{Z}_p$ when $\mathrm{Char}(\mathbb{F}) \nmid d$:

- $\mathcal{O}(d^2 + \delta\, d\, \log(p))$ $\leftarrow$ was $\mathcal{O}(d^2 + \delta\, d\, \log(p) + d\delta^2)$!

Practical aspects:

- Sage prototype,
- some `C` implementation via Flint (irreducibility for $\mathbb{A} = \mathbb{F}_q[[t]]$).

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