Reduction point algorithm for Fuchsian groups

Dionís Remón

Universitat de Barcelona

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Reduction point algorithm

Definition

Given a pair $(\Gamma, \mathcal{F}(\Gamma))$ and a point $z_0 \in \mathcal{H}$, the *reduction point algorithm problem* asks for an explicit transformation $\gamma \in \Gamma$ such that $\gamma(z_0) \in \mathcal{F}(\Gamma)$.

Word problem

Definition

The *word problem* for a finitely generated group G is the algorithmic problem of deciding whether two words in the generators of G represent the same element.

Example

Let G be a group generated by a set of elements $\{\gamma_2, \gamma_4, \gamma_6\}$ with relations $\gamma_2^3 = \gamma_4^3 = \gamma_6^2 = \mathrm{Id}$. Is $\gamma_2^{-1} = \gamma_2^2$ true?

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Weak word problem

The *weak word problem algorithm* for a finitely generated fuchsian group G and fixed $\mathcal{F}$, i.e., fixed a presentation (generators and relations) is the problem of writing explicitly an element $g \in G$ in terms of its generators.

Equivalence of the two problems

Theorem

Let Γ be a Fuchsian group and $\mathcal{F}(\Gamma)$ a fundamental domain for Γ. The word problem algorithm and the reduction point algorithm are equivalent.

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Proof.

Let $g \in \Gamma$ be an element that we want to describe by using a set of generators of Γ. Let $z_0 \in \overset{\circ}{\mathcal{F}}$ and $z = g(z_0)$. We observe that if $g \neq \mathrm{Id}$, then $z \notin \mathcal{F}$. Applying the reduction point algorithm, we obtain $\gamma(z) = z^* \in \mathcal{F}$. This equality means, by the uniqueness of equivalent points, that $z_0 = z^*$. Then $z_0 = \gamma(z) = \gamma(g(z_0))$. We deduce that $\gamma.g = \mathrm{Id}$. This leads to $g = \gamma^{-1}$. This solves the word problem, since we know how to write γ in terms of the generators of Γ.

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Motivation

Why is the reduction algorithm interesting?

Maass waveforms and modular forms

Definition

A Maass waveform for a Fuchsian group Γ is a function $f : \mathcal{H} \to \mathbb{C} \cup \{\infty\}$, infinitely differentiable and such that:

- It is en eigenvector for the hyperbolic Laplacian: $\Delta f = \lambda f$ with

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} \right),$$

  where we understand $z = x + iy$.
- The function f satisfies the cuspidality condition

$$\int_{\mathcal{F}} |f(z)|^2 ds^2 < \infty.$$

- $f(\gamma(z)) = \chi(\gamma) f(z)$, with $\gamma \in \Gamma$.

A Maass-wave form admits a series development of the form

$$f(z) = \sum_{n=-\infty}^{n=\infty} a_n \sqrt{\Im(z)} K_{i\mu}(2\pi |n| \Im(z)) e^{2\pi i n \Re(z)},$$

where $K_{i\mu}$ are modified Bessel functions.

To compute the coefficients a_n we need to solve a linear system

$$\begin{aligned} f_j(z) &= f(\sigma_j z) = f(T_j^{-1} U_{w_j}^{-1} \sigma_{I(j)} z_j^*) \\ &= \chi(T_j^{-1} U_{w_j}^{-1}) f_{I(j)}(z_j^*). \end{aligned} \tag{1}$$

for $j = 1, \ldots, n$, with $z \in \mathcal{H}$, $\gamma(z) = z^*$, and $z^* \in \mathcal{F}$. The number n equals the number of auxiliary points $\{z_i\} \subset \mathcal{H}$, necessary to compute the coefficients.

Fuchsian codes

The reduction algorithm can be used to design codes; see Iván Blanco-Chacón's tuesday talk.

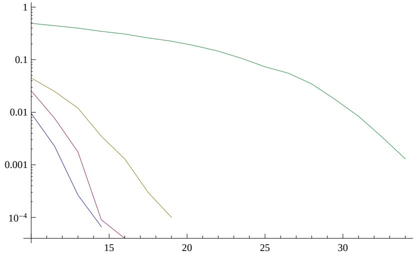

Figure: "4x2-NUPAM" Symbols

$\Gamma(6,1)$, Alsina-Bayer

$\Gamma(6,1)$

Then the hyperbolic hexagon of vertices

$$v_1 = \frac{-\sqrt{3}+i}{2}, \quad v_2 = \frac{-1+i}{1+\sqrt{3}}, \quad v_3 = (2-\sqrt{3})i,$$
$$v_4 = \frac{1+i}{1+\sqrt{3}}, \quad v_5 = \frac{\sqrt{3}+i}{2}, \quad v_6 = i,$$

is a fundamental domain for $\Gamma(6,1)$ in the Poincarè upper half-plane.

The transformations which fix the vertices ($\gamma_i(v_i) = v_i$) are:

$$\gamma_1 = \begin{bmatrix} \sqrt{3} & 2 \\ -2 & -\sqrt{3} \end{bmatrix}, \qquad \gamma_2 = \frac{1}{2}\begin{bmatrix} 1+\sqrt{3} & 3-\sqrt{3} \\ -3-\sqrt{3} & 1-\sqrt{3} \end{bmatrix},$$

$$\gamma_3 = \begin{bmatrix} 0 & -2+\sqrt{3} \\ 2+\sqrt{3} & 0 \end{bmatrix}, \qquad \gamma_4 = \frac{1}{2}\begin{bmatrix} 1+\sqrt{3} & -3+\sqrt{3} \\ 3+\sqrt{3} & 1-\sqrt{3} \end{bmatrix},$$

$$\gamma_5 = \begin{bmatrix} \sqrt{3} & -2 \\ 2 & -\sqrt{3} \end{bmatrix}, \qquad \gamma_6 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

$\Gamma(6,1)$, Alsina-Bayer

Presentation of $\Gamma(6,1)$

We have the following presentation of the group $\Gamma(6,1)/(\pm\mathrm{Id})$:

$$\langle \gamma_2, \gamma_4, \gamma_6 : \gamma_2^3 = \gamma_4^3 = \gamma_6^2 = (\gamma_2^{-1}\gamma_6\gamma_4)^2 = 1 \rangle.$$

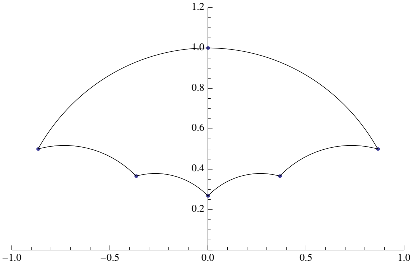

Figure: Fundamental domain $\mathcal{F}(\Gamma(6,1))$

Ingredients of a reduction point algorithm

We can cover $\mathcal{H}$ by "regions", depending on $\mathcal{F}(\Gamma(6,1))$, and assign a map to each region so that

- The regions are finite in number and satisfy $\mathcal{H} = \bigcup \mathcal{R}_i$.
- Do mappings $(\mathcal{R}_i, \gamma_i)$.
- In each region, the reduction algorithm uses words beginning with γ_i^{-1}.

Ingredients of a reduction point algorithm (2)

We are going to show

- A mapping assignment for the Fuchsian group $\Gamma(6,1)$ is given by (S^-, γ_2), (S^+, γ_4), (S^∞, γ_6) and $(\mathcal{F}, \mathrm{Id})$.

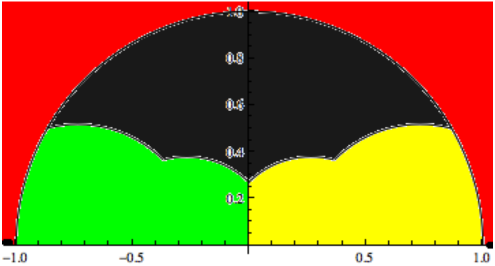

Figure: Regions for $\mathcal{F}(\Gamma(6,1))$

Proof

Idea of the proof

We shall show that all points of each region can be reach by using in the last position (as a application) the paired map of the region.

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Region S^∞

Every point in the region S^∞ can be moved to one of the other regions via the map γ_6. This is because every point $z \in \mathcal{H}$ with $|z| > 1$ via the inversion γ_6 is translated to $|\gamma_6(z)| < 1$.

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Region S^- and S^+

In what follows we are going to work in the region S^-. The same procedure can be applied for the region S^+.

Proof (ii)

Boundary of S^-

We consider the boundary of S^- composed by three "edges". These are: S_u (upper edge), S_e (exterior edge) and S_i (interior edge).

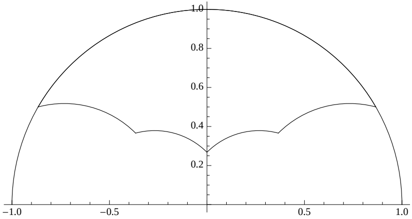

Figure: Edges

Next

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Difficult one

To cover the interior edge S_i is more difficult because the tiles which cover the edge S_i intersect the region S^+.

Exterior edge

Let p be the path which contains the vertices $v_1 v_6 v_5$. Then the map $(\gamma_2\gamma_4^2)$ satisfies

$$\bigcup_{n\in\mathbb{N}} (\gamma_2\gamma_4^2)^n(p) = \{z : |z| = 1, \Re(z) < 0\}.$$

This means that the (infinite) set of tiles $(\gamma_2\gamma_4^2)^n(\mathcal{F})$ covers S_e^-.

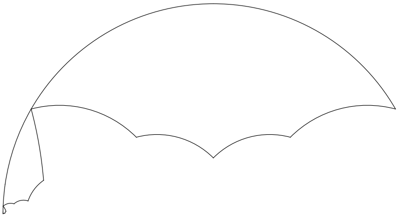

Figure: Subregions for S^-

Upper edge

The upper edge of S^-, i. e. S_u^-, is covered by three tiles

$$\gamma_2(\mathcal{F}) \cup \gamma_2^2(\mathcal{F}) \cup \gamma_2^2\gamma_6(\mathcal{F}).$$

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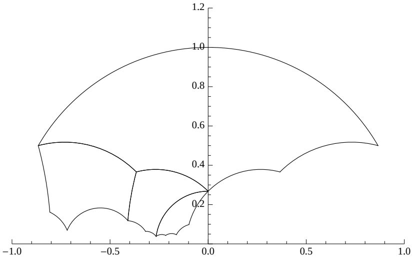

Figure: Higher edge for S^-

Interior edge

The interior edge S_i^- of S^- is covered by the set of infinite tiles

$$\mathcal{E} := \bigcup_{n\geq 0} \gamma_2^2\gamma_6\gamma_4 h^n(\mathcal{F}) \cup \bigcup_{n>0} h^{-n}(\mathcal{F}).$$

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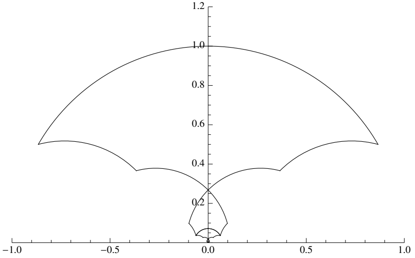

Figure: Covering the interior edge of S^-

From the presentation of the group $\Gamma(6,1)$, it follows that

$$\gamma_2^2\gamma_6\gamma_4 = \gamma_4^2\gamma_6\gamma_2.$$

This means, from the word problem point of view, that the region $\mathcal{E}$, which includes S_i, corresponds to the set of words that can be written starting either with γ_2 or γ_4.

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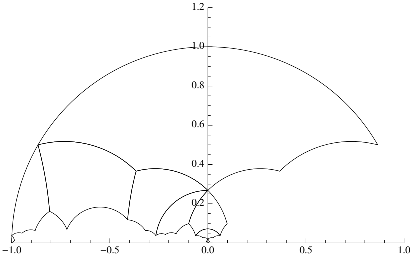

Figure: Covering the interior of S^-

The star $\star(\mathcal{F})$ of $\mathcal{F}$

Covering a fundamental domain

Let Γ be a fuchsian group with fundamental domain $\mathcal{F}$. Let $\mathcal{T} \subseteq \Gamma$ be a set of transformations of Γ. We say that $\mathcal{T}$ covers $\mathcal{F}$ if the set

$$\mathcal{C}_{\mathcal{T}} := \bigcup_{\gamma \in \mathcal{T}} \gamma(\mathcal{F})$$

is connected and there exists an $\epsilon > 0$ such that $B_{\epsilon}(z) \subseteq \mathcal{C}_{\mathcal{T}}$, for all in $z \in \mathcal{H}$.

Star of $\mathcal{F}$

We define $\Gamma^\star \subseteq \Gamma$ as the set of transformations such that

$$\bigcup_{\gamma \in \Gamma^*} \gamma(\mathcal{F}) = \mathcal{C}_{\Gamma^*} = \bigcap_{\mathcal{T}} \mathcal{C}_{\mathcal{T}}.$$

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We define $\star(\gamma_1(\mathcal{F}) \cup \gamma_2(\mathcal{F})) := \star(\gamma_1(\mathcal{F})) \cup \star(\gamma_2(\mathcal{F}))$.

Properties of $\star(\mathcal{F})$

P.1

Let $\mathcal{F}$ be a fundamental domain for Γ and $\gamma \in \Gamma$. Then, $\star(\gamma\mathcal{F})) = \gamma(\star(\mathcal{F}))$.

P.2

Let $\gamma \in \Gamma(6,1)$ be such that $\gamma(\mathcal{F}) \subseteq S^- \setminus \partial S^-$, i. e., with no intersection with the boundary. Then,

$$\star(\gamma(\mathcal{F})) \subseteq S^- \cup \mathcal{E}.$$

Properties of $\star(\mathcal{F})$ (cont.)

P.3

Let Γ be a Fuchsian group and let $\mathcal{F}$ be a fundamental domain for Γ. We have the following property for the star operator:

$$\inf\{\Im(z) : z \in \gamma(\mathcal{F})\} > \inf\{\Im(z) : z \in \star(\gamma(\mathcal{F}))\},$$

for all $\gamma \in \Gamma$.

Example, $\Gamma(6,1)$

The set $\Gamma(6,1)^*$ is given by

$$\Gamma(6,1)^* = \{\, \mathrm{Id}, \gamma_2, \gamma_4, \gamma_6, \gamma_2^2, \gamma_4^2, \gamma_6\gamma_2, \gamma_6\gamma_4, \gamma_2\gamma_4^2, \gamma_4\gamma_2^2, \gamma_2^2\gamma_6,$$
$$\gamma_4^2\gamma_6, \gamma_6\gamma_4\gamma_2^2, \gamma_2^2\gamma_6\gamma_4, \gamma_6\gamma_2\gamma_4^2 \,\}.$$

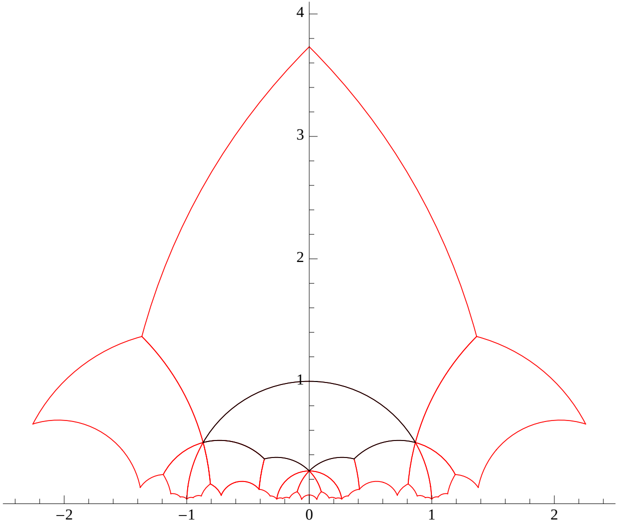

Figure: $\star(\mathcal{F}(\Gamma(6,1)))$

In order to reach the result, we must cover the region S^-. Once we control the edges we must cover the rest.

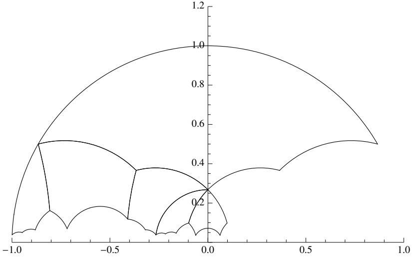

Figure: S_1^-

We define

$$\begin{aligned} S_1^- &:= \star(\mathcal{F}) \cap (S^- \cup \mathcal{E}), \text{then} \\ S_1^- &= \gamma_2\gamma_4^2(\mathcal{F}) \cup \gamma_2(\mathcal{F}) \cup \gamma_2^2(\mathcal{F}) \cup \gamma_2^2\gamma_6(\mathcal{F}) \cup \gamma_2^2\gamma_6\gamma_4(\mathcal{F}) \\ b_1 &:= \inf\{\Im(z) : z \in S_1^-\} \end{aligned}$$

$$N_{21}^{-} = (\star(S_1^{-}) \setminus S_1^{-}) \cap (S^{-} \cup \mathcal{E}),$$
$$t_{21} = \sup\{\Im(z) : z \in N_{21}^{-}\}$$

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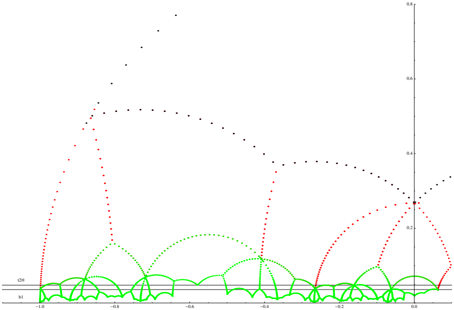

Figure: N_{21}^{-} for the $\Gamma(6, 1)$

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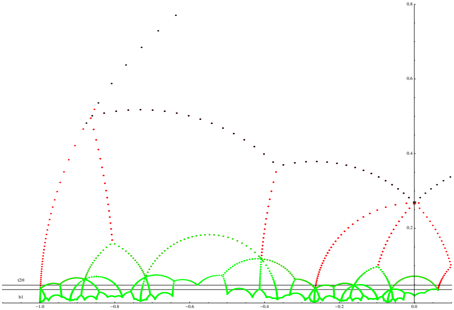

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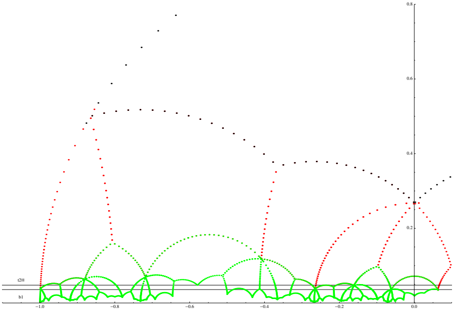

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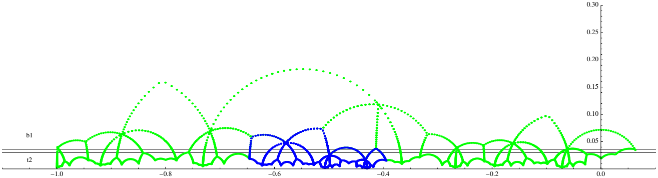

Figure: S_2^- for $\Gamma(6,1)$

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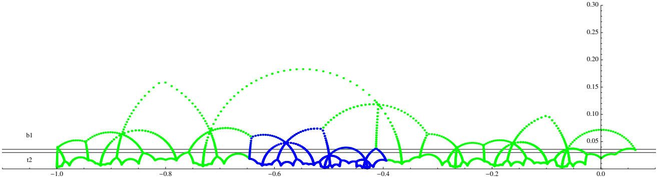

Figure: S_2^- for $\Gamma(6,1)$

We name t_2 the last t_{2x} and we name S_2 the union of the green plus blues tiles.

Main theorem

Theorem

We have the equality:

$$S^- = \bigcup_{n \in \mathbb{N}} S_n^- .$$

Proof.

We have constructed a sequence $\{t_i > 0\}$ which is strictly decreasing

$$t_1 > t_2 > t_3 > \ldots$$

This means that $\lim t_i = 0$ and this implies that S^- can be covered by tiles. The property $P.1$ implies that the region can be covered by maps, starting as a word, by g_2. □

Reduction algorithm

Correctness

The algorithm finishes always since the action of the group Γ on $\mathcal{H}$ is properly discontinue (discretness).

Pairing Map-Region

element of $\Gamma(6,1)$	Regions
γ_2	S^-
γ_4	S^+
γ_6	S^∞

```
posaDomini612[z_] :=
 Block[{bandera, bandera1, bandera2, bandera3, bandera4, c1, r1, c2, r2,
   c3, r3, c4, r4, v1, v2, v3, v4, v5, aux, k = 0, g1, g2, g3, g4, g5, g6},
  aux = z;
  v1 = -Sqrt[3] / 2 + I / 2; v2 = (-1 + I) / (1 + Sqrt[3]);
  v3 = (2 - Sqrt[3]) I; v4 = (1 + I) / (1 + Sqrt[3]); v5 = (Sqrt[3] + I) / 2;
  c1 = retornaCentre[v1, v2]; r1 = Abs[v2 - c1];
  c2 = retornaCentre[v2, v3]; r2 = Abs[v3 - c2];
  c3 = retornaCentre[v3, v4]; r3 = Abs[v4 - c3];
  c4 = retornaCentre[v4, v5]; r4 = Abs[v5 - c4];
  g2 = 1 / 2 {{1 + Sqrt[3], 3 - Sqrt[3]}, {-3 - Sqrt[3], 1 - Sqrt[3]}};
  g4 = 1 / 2 {{1 + Sqrt[3], -3 + Sqrt[3]}, {3 + Sqrt[3], 1 - Sqrt[3]}};
  g6 = {{0, 1}, {-1, 0}};
  While[ ! esenDomini[{{0, 1}}, {{c1, r1}, {c2, r2}, {c3, r3}, {c4, r4}}, aux],
   If[Abs[aux] ≥ 1, aux = hm2[g6, aux],
    If[Re[aux] ≤ 0 , aux = hm2[g2, aux], aux = hm2[g4, aux]]];
   k++;
   If[k > 1000, Abort[]]
  ];
  Return[aux]
 ]
```

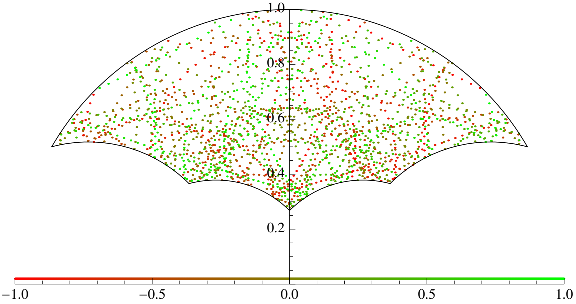

Figure: Example of the use of our reduction algorithm

$\Gamma(10,1)$, Alsina-Bayer

$$v_1 = \frac{-\sqrt{2}+\sqrt{3}i}{5(-1+\sqrt{2})}, \quad v_2 = \frac{-\sqrt{2}+\sqrt{3}i}{5(1+\sqrt{2})} \quad v_3 = \frac{-\sqrt{2}+\sqrt{3}i}{5(7+5\sqrt{2})}i$$

$$v_4 = \frac{\sqrt{2}+\sqrt{3}i}{5(7+5\sqrt{2})}, \quad v_5 = \frac{\sqrt{2}+\sqrt{3}i}{5(1+\sqrt{2})} \quad v_6 = \frac{\sqrt{2}+\sqrt{3}i}{5(-1+\sqrt{2})},$$

The transformations which fixe the vertices ($\gamma_i(v_i) = v_i$) are:

$$\gamma_1 = \frac{1}{2}\begin{bmatrix} 1+\sqrt{2} & 1+\sqrt{2} \\ 5(1-\sqrt{2}) & 1-\sqrt{2} \end{bmatrix} \qquad \gamma_2 = \frac{1}{2}\begin{bmatrix} 1+\sqrt{2} & -1+\sqrt{2} \\ -5(1+\sqrt{2}) & 1-\sqrt{2} \end{bmatrix}$$

$$\gamma_3 = \frac{1}{2}\begin{bmatrix} 1+\sqrt{2} & -7+5\sqrt{2} \\ -5(7+5\sqrt{2}) & 1-\sqrt{2} \end{bmatrix} \qquad \gamma_4 = \frac{1}{2}\begin{bmatrix} 1+\sqrt{2} & 7-5\sqrt{2} \\ 5(7+5\sqrt{2}) & 1-\sqrt{2} \end{bmatrix}$$

$$\gamma_5 = \frac{1}{2}\begin{bmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 5(1+\sqrt{2}) & 1-\sqrt{2} \end{bmatrix} \qquad \gamma_6 = \frac{1}{2}\begin{bmatrix} 1+\sqrt{2} & -1-\sqrt{2} \\ 5(-1+\sqrt{2}) & 1-\sqrt{2} \end{bmatrix}$$

The transformations which fixe the vertices ($\gamma_i(v_i) = v_i$) are:

$$\gamma_1 = \frac{1}{2}\begin{bmatrix} 1+\sqrt{2} & 1+\sqrt{2} \\ 5(1-\sqrt{2}) & 1-\sqrt{2} \end{bmatrix} \qquad \gamma_2 = \frac{1}{2}\begin{bmatrix} 1+\sqrt{2} & -1+\sqrt{2} \\ -5(1+\sqrt{2}) & 1-\sqrt{2} \end{bmatrix}$$

$$\gamma_3 = \frac{1}{2}\begin{bmatrix} 1+\sqrt{2} & -7+5\sqrt{2} \\ -5(7+5\sqrt{2}) & 1-\sqrt{2} \end{bmatrix} \qquad \gamma_4 = \frac{1}{2}\begin{bmatrix} 1+\sqrt{2} & 7-5\sqrt{2} \\ 5(7+5\sqrt{2}) & 1-\sqrt{2} \end{bmatrix}$$

$$\gamma_5 = \frac{1}{2}\begin{bmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 5(1+\sqrt{2}) & 1-\sqrt{2} \end{bmatrix} \qquad \gamma_6 = \frac{1}{2}\begin{bmatrix} 1+\sqrt{2} & -1-\sqrt{2} \\ 5(-1+\sqrt{2}) & 1-\sqrt{2} \end{bmatrix}$$

Principal homothety

$$h = \begin{bmatrix} 3+2\sqrt{2} & 0 \\ 0 & 3-2\sqrt{2} \end{bmatrix}.$$

Presentation of $\Gamma(10,1)$

$$\langle \gamma_2, h, \gamma_5 : \ \gamma_2^3 = \gamma_5^3 = (h^{-1}\gamma_2)^3 = (h^{-1}\gamma_5)^3 = 1\rangle.$$

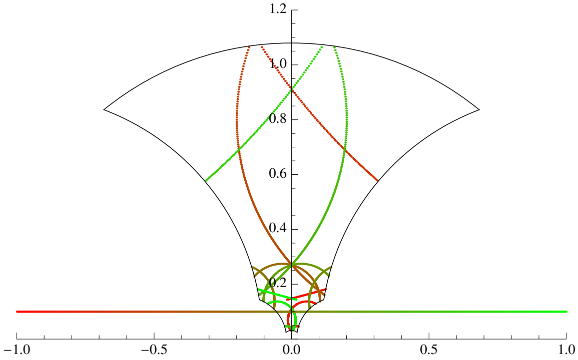

Figure: Example of the use of our reduction algorithm

$\Gamma(15,1)$, Alsina-Bayer

The principal homothety of $\Gamma(15,1)$ is

$$h = \begin{bmatrix} 2+\sqrt{3} & 0 \\ 0 & 2-\sqrt{3} \end{bmatrix}.$$

$$\beta = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix}, \quad \gamma = \frac{1}{2}\begin{bmatrix} -4+3\sqrt{3} & -\sqrt{3} \\ 5\sqrt{3} & -4-3\sqrt{3} \end{bmatrix}$$

Presentation of the group

$$\Gamma(15,1)/\{\pm\mathrm{Id} = \langle h, \beta, \gamma : (\gamma h)^3 = (h\beta^{-1}\gamma\beta)^3 = 1\rangle.$$

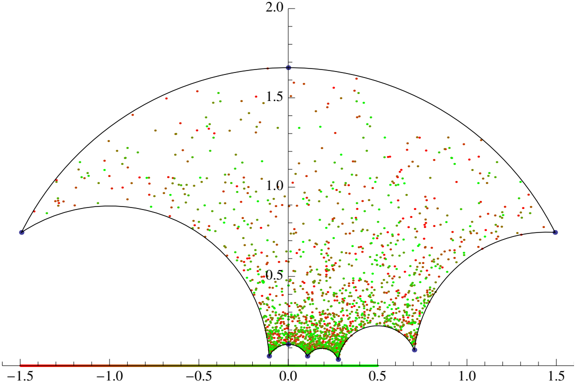

Figure: Example of the use of our algorithm

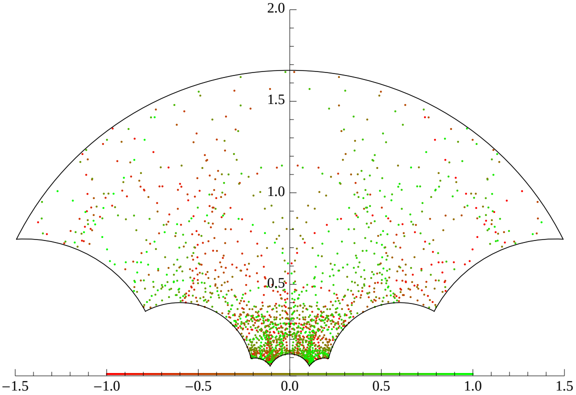

Figure: Example of the use of our algorithm

$\Gamma(6,5)$, Nualart-Travesa

A fundamental domain for $\Gamma(6,5)$ is defined by the vertices

$$v_1 = (2+\sqrt{3}i), \quad v_2 = \frac{-2\sqrt{3}+i}{4+\sqrt{3}}, \quad v_3 = \frac{16\sqrt{3}+i}{38+15\sqrt{3}}, \quad v_4 = \frac{-15\sqrt{3}+i}{38+16\sqrt{3}}, \quad v_5 = \frac{-\sqrt{3}+i}{4+2\sqrt{3}},$$

$$v_6 = (7-4\sqrt{3})i, \quad v_7 = \frac{2\sqrt{3}+i}{5+2\sqrt{3}}, \quad v_8 = \frac{16\sqrt{3}+i}{31+8\sqrt{3}}, \quad v_9 = \frac{15\sqrt{3}+i}{28+6\sqrt{3}}, \quad v_{10} = \frac{\sqrt{3}+i}{2}.$$

$$g_1 = \begin{bmatrix} 0 & -2-\sqrt{3} \\ 2-\sqrt{3} & 0 \end{bmatrix}, \qquad g_2 = \begin{bmatrix} -2\sqrt{3} & -4+\sqrt{3} \\ 4+\sqrt{3} & 2\sqrt{3} \end{bmatrix},$$

$$g_3 = \begin{bmatrix} 16\sqrt{3} & 38-15\sqrt{3} \\ -38-15\sqrt{3} & -16\sqrt{3} \end{bmatrix}, \qquad g_4 = \begin{bmatrix} -15\sqrt{3} & -38+16\sqrt{3} \\ 38+16\sqrt{3} & 15\sqrt{3} \end{bmatrix},$$

$$g_5 = \begin{bmatrix} \sqrt{3} & 4-2\sqrt{3} \\ -4-2\sqrt{3} & -\sqrt{3} \end{bmatrix}, \qquad g_6 = \begin{bmatrix} 0 & 7-4\sqrt{3} \\ -7-4\sqrt{3} & 0 \end{bmatrix},$$

$$g_7 = \begin{bmatrix} -2\sqrt{3} & 5-2\sqrt{3} \\ -5-2\sqrt{3} & 2\sqrt{3} \end{bmatrix}, \qquad g_8 = \begin{bmatrix} -16\sqrt{3} & 31-8\sqrt{3} \\ -31-8\sqrt{3} & 16\sqrt{3} \end{bmatrix},$$

$$g_9 = \begin{bmatrix} -15\sqrt{3} & 28-6\sqrt{3} \\ -28-6\sqrt{3} & 15\sqrt{3} \end{bmatrix}, \qquad g_{10} = \begin{bmatrix} \sqrt{3} & -2 \\ 2 & -\sqrt{3} \end{bmatrix}.$$

Presentation of $\Gamma(6,5)$

Identification of sides

The γ_i, $i = 1, \ldots, 5$,

$$\begin{aligned}
&\text{the map } \gamma_1 \text{ sends } (v_1 v_2, v_7 v_6),\\
&\text{the map } \gamma_2 \text{ sends } (v_2 v_3, v_8 v_7),\\
&\text{the map } \gamma_3 \text{ sends } (v_3 v_4, v_1 v_{10}),\\
&\text{the map } \gamma_4 \text{ sends } (v_4 v_5, v_{10} v_9),\\
&\text{the map } \gamma_5 \text{ sends } (v_5 v_6, v_9 v_8).
\end{aligned}$$

Presentation of $\Gamma(6,5)$

$$\Gamma(6,5) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 : (\gamma_3 \gamma_2^{-1} \gamma_1)^2 = (\gamma_2^{-1} \gamma_5 \gamma_1)^2 = (\gamma_4^{-1} \gamma_5)^2 = \mathrm{Id}\}.$$

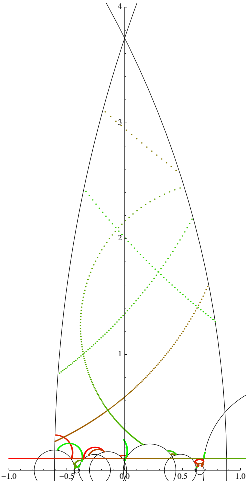

Figure: Example of the use of our reduction algorithm

The triangle group $\Gamma = e2d1D6ii$, Sijsling

Consider the Fuchsian triangle group $\Gamma = e2d1D6ii$ which has signature $(1;2)$. Then the hyperbolic polygon with vertices (v_1, v_2, v_3, v_4),

$$v_1 = \frac{1}{2}i\sqrt{2+\sqrt{3}} + \frac{1}{2}\sqrt{3\left(2+\sqrt{3}\right)}, \quad v_2 = \frac{1}{2}\sqrt{6-3\sqrt{3}} + \frac{1}{2}i\sqrt{2-\sqrt{3}},$$

$$v_3 = \frac{1}{2}i\sqrt{2+\sqrt{3}} - \frac{1}{2}\sqrt{3\left(2+\sqrt{3}\right)}, \quad v_4 = \frac{1}{2}i\sqrt{2-\sqrt{3}} + \frac{-3+\sqrt{3}}{2\sqrt{2}},$$

is a fundamental domain in the Poincarè upper half-plane.

...

Moreover we have the next properties:

- Let α, β be maps

$$\alpha = \begin{bmatrix} \sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}} \end{bmatrix} \quad \beta = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}.$$

- The identifications between edges are given by:

$$(v_3 v_4, v_1 v_2) \text{ by means of } \beta,$$
$$(v_2 v_4, v_1 v_3) \text{ by means of } \alpha.$$

- We have the next presentation for this group:

$$\langle \alpha, \beta \ : \ (\alpha\beta\alpha^{-1}\beta^{-1})^2 = 1 \rangle.$$

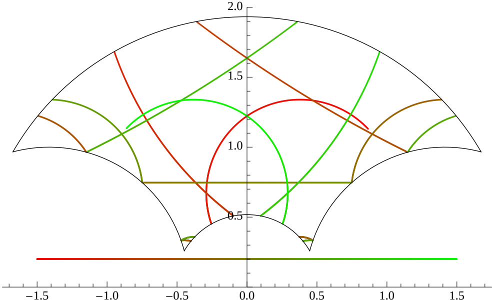

Figure: Example of the use of the reduction algorithm