JACOBIANS OF MUMFORD CURVES. A NEW PERSPECTIVE FOR GENERALIZATIONS

Abstract. Our goal is to give an idea of the construction of the Jacobian of an analytic Mumford curve in the Berkovich sense over any complete non-archimedean field.

After Mumford introduced the homonymous curves in his famous paper published in 1972 ([Mum72]), Manin and Drinfeld built their Jacobian in the local case ([MD73]) and Gerritzen and van der Put when the ground field is assumed algebraically closed ([GvdP80]), always by means of theta functions. More recently, several works by Darmon and Dasgupta between others showed that those function could be seen in some way as mutiplicative integrals, defined in all their generality by Longhi. This let us to enlighten on the construction of these abelian varieties over any complete non-archimedean ground field and through the Berkovich analytic theory.

Let K be a complete non-archimedean field with absolute value $|\cdot|$ and valuation $v_K(\cdot) = -\log|\cdot|$. Let $\Gamma \subset \mathrm{PGL}_2(K)$ be a Schottky group, let $\mathcal{L} = \mathcal{L}_\Gamma \subset {\mathbb{P}^1_K}^*(K)$ be its set of limit points and let $\mathcal{L}^* \subset \mathbb{P}^1_K(K)$ be the set of corresponding points of the points of $\mathcal{L}$. The analytic Mumford curve associated to Γ is the quotient of the analytic space $\Omega_\mathcal{L} := ({\mathbb{P}^1_K}^*)^{an} \setminus \mathcal{L}$ by the group, that is

$$C_\Gamma := \Omega_\mathcal{L}/\Gamma.$$

Then, given an arbitrary point $p \in \Omega_\mathcal{L}(K)$, the Jacobian of C_Γ is defined by means of the map

$$\begin{array}{ccc} \Gamma^{ab} & \xrightarrow{\;\;\times\!\!\!\!\!\!\int_{\cdot} d\;\;} & \mathrm{Hom}(\mathcal{M}(\mathcal{L}^*, \mathbb{Z})_0^\Gamma, K^*) \\ \gamma & \longmapsto & \left\{ \times\!\!\!\!\!\!\int_{\gamma p - p} d : \mu \mapsto \times\!\!\!\!\!\!\int_{\gamma p - p} d\mu \right\} \end{array}$$

as follows: writing $T := \mathrm{Hom}(\mathcal{M}(\mathcal{L}^*, \mathbb{Z})_0^\Gamma, K^*)$ and $\Lambda := \mathrm{Im}\left(\times\!\!\!\!\!\!\int_{\cdot} d\right)$, we have

$$\mathrm{Jac}(C_\Gamma) \cong T/\Lambda.$$

We get the Abel-Jacobi map choosing any point $z_0 \in C_\Gamma(K)$ and applying integration:

$$\begin{array}{ccc} C_\Gamma(K) & \xrightarrow{\;\;i_{z_0}\;\;} & T/\Lambda(K) \\ z & \longmapsto & \times\!\!\!\!\!\!\int_{z - z_0} d \end{array}$$

In order to get the exposed above we divide the proof in two parts.

2010 *Mathematics Subject Classification.* Primary: 14H40; Secondary 14G22.

Key words and phrases. Jacobian, Mumford curve, Bruhat-Tits tree, analytic geometry, multiplicative integral, harmonic measure.

(1) Build the map $\fint_{\cdot} d$ and prove that T/Λ is an abelian variety.
(2) Prove that in fact $T/\Lambda \cong \mathrm{Div}_0(C_\Gamma)/\operatorname{Prin}(C_\Gamma) \cong \operatorname{Jac}(C_\Gamma)$.

Through this work, the non-archimedean analytic theory we deal with is the Berkovich one, instead of the rigid analytic.

We define the analytic projective line $({\mathbb{P}^1_K}^*)^{an}$ as the set of multiplicative seminorms on the polynomial ring $K[X_0, X_1]$ extending $|\cdot|$ modulo a certain equivalence relation.

Its open skeleton $\mathcal{T}_K$ coincides with the set of norms on K^2 diagonalizable with respect to some basis modulo homothety, which is the Bruhat-Tits building of $\mathrm{PGL}_2(K)$. It is a metric space. We call its "compactification" $\overline{\mathcal{T}_K}$ to the set of seminorms on K^2 diagonalizable with respect to some basis modulo homothety (but it is compact only when K is locally compact). Identifying $K^2 \cong KX_0 + KX_1$ we have an embedding $\overline{\mathcal{T}_K} \hookrightarrow ({\mathbb{P}^1_K}^*)^{an}$. Then, here we have also $\overline{\mathcal{T}_K} \setminus \mathcal{T}_K = {\mathbb{P}^1_K}^*(K) \subset ({\mathbb{P}^1_K}^*)^{an}$.

We can identify the points of $\overline{\mathcal{T}_K}$ with balls $B(x, r)$ of radii r around points $x \in {\mathbb{P}^1_K}^*(K)$, which correspond to seminorms $\alpha(x, r)$. The norms are the ones with $r > 0$, so $\overline{\mathcal{T}_K} \setminus \mathcal{T}_K$ are those with $r = 0$.

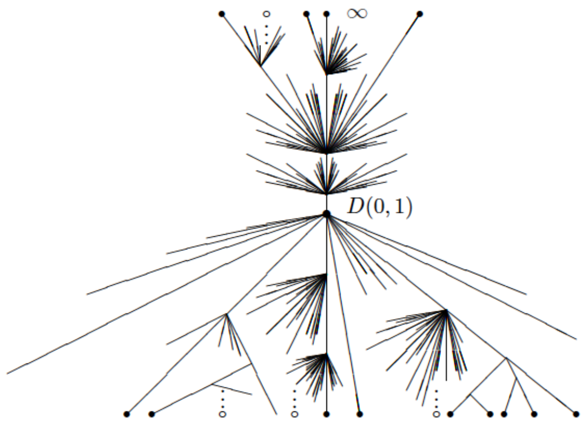

FIGURE 1. The Berkovich projective line (illustration got from the book [BR10])

Given two points $x_0, x_1 \in {\mathbb{P}^1_K}^*(K)$, the apartment $\mathbb{A}_{x_0,x_1}$ is the open path inside $\mathcal{T}_K$ between $\alpha(x_0, 0)$ and $\alpha(x_1, 0)$. It is isometric with $\mathbb{R}$.

Let $\mathcal{L} \subset {\mathbb{P}^1_K}^*(K)$ be a compact set with at least two points. We define its associated tree by

$$\mathcal{T}_K(\mathcal{L}) := \bigcup_{x_0,x_1 \in \mathcal{L}} \mathbb{A}_{x_0,x_1}.$$

It is locally finite.

We have a retraction map

$$\mathrm{r}_{\mathcal{L}} : ({\mathbb{P}^1_K}^*)^{an} \longrightarrow \overline{\mathcal{T}_K(\mathcal{L})}$$

An edge in $\mathcal{T}_K(\mathcal{L})$ is a path of the tree without vertices of valence greater than three in it and of finite length -since it has no vertices it is inside some apartment $\mathbb{A}$ and by means of the isometry we can transport the distance on $\mathbb{R}$ to $\mathbb{A}$-.

Given an edge e in $\mathcal{T}_K(\mathcal{L})$ we associate to it an open compact set $\mathcal{B}(e) \subset \mathcal{L}^*$. These sets form a basis for the topology of the compact space $\mathcal{L}^*$.

These conditions let us define an harmonic measure ($\mathbb{Z}$-valued) on $\mathcal{L}^*$ as a map μ on the set of open compact subsets of $\mathcal{L}^*$ with integer values, such that it is additive on disjoint subsets ($\mu(\mathcal{U} \sqcup \mathcal{V}) = \mu(\mathcal{U}) + \mu(\mathcal{V})$) and it vanishes on the total space ($\mu(\mathcal{L}^*) = 0$; this is the harmonicity condition). We denote the set of harmonic measures on $\mathcal{L}^*$ by $\mathcal{M}(\mathcal{L}^*, \mathbb{Z})_0$.

Since the sets $\mathcal{B}(e)$ are a basis for the topology of $\mathcal{L}^*$, the values $\mu(\mathcal{B}(e))$ determine the measure. By its identification with harmonic measures we shall write $\mu(e) := \mu(\mathcal{B}(e))$.

Next let G be a complete topological abelian group (such as K^* or $\mathbb{R}_{>0}$). Given a function $f : \mathcal{L}^* \longrightarrow G$ and $\mu \in \mathcal{M}(\mathcal{L}^*, \mathbb{Z})_0$ we define the multiplicative integral as the limit

$$\times\!\!\!\!\!\!\int_{\mathcal{L}^*} f d\mu := \lim_{\mathscr{C}} \prod_{t \in \mathcal{U} \in \mathscr{C}} f(t)^{\mu(\mathcal{U})}$$

where it is taken on coverings $\mathscr{C}$ of $\mathcal{L}^*$ by open compacts, the products are over all the opens $\mathcal{U} \in \mathscr{C}$ and the points $t \in \mathcal{U}$ are arbitrary.

Given a divisor $D \in \mathrm{Div}_0(\Omega_{\mathcal{L}}(K))$ we build a function $f_D : \mathcal{L}^* \longrightarrow K^*$ in such a way that the multiplicative integral of this function depens only on D, so we get a map

$$\begin{array}{ccc} \mathrm{Div}_0(\Omega_{\mathcal{L}}) & \xrightarrow{\times\!\!\!\!\!\int_{\cdot} d} & \mathrm{Hom}(\mathcal{M}(\mathcal{L}^*, \mathbb{Z})_0, K^*) \\ D & \longmapsto & \left\{ \times\!\!\!\!\!\!\int_D d : \mu \mapsto \times\!\!\!\!\!\!\int_D d\mu := \times\!\!\!\!\!\!\int_{\mathcal{L}^*} f_D d\mu \right\} \end{array}$$

Given a divisor $D \in \mathrm{Div}_0(\mathcal{T}_K(\mathcal{L}))$ we build a function $\varphi_D : \mathcal{L}^* \longrightarrow \mathbb{R}_{>0}$ in such a way that the multiplicative integral of this function depends only on D, so we get a map

$$\begin{array}{ccc} \mathrm{Div}_0(\mathcal{T}_K(\mathcal{L})) & \xrightarrow{\left|\times\!\!\!\!\!\int\right|_{\cdot} d} & \mathrm{Hom}(\mathcal{M}(\mathcal{L}^*, \mathbb{Z})_0, \mathbb{R}_{>0}) \\ D & \longmapsto & \left\{ \left|\times\!\!\!\!\!\!\int\right|_D d : \mu \mapsto \left|\times\!\!\!\!\!\!\int\right|_D d\mu := \times\!\!\!\!\!\!\int_{\mathcal{L}^*} \varphi_D d\mu \right\} \end{array}$$

Lemma 1. *Given $D \in \mathrm{Div}_0(\Omega_{\mathcal{L}}(K))$ we have*

$$\left| \times\!\!\!\!\!\!\int_D d\mu \right| = \left| \times\!\!\!\!\!\!\int \right|_{\mathrm{r}_{\mathcal{L}}(D)} d\mu$$

Now, we consider a Schottky group Γ and $\mathcal{L} = \mathcal{L}_\Gamma$ as in the introduction. Applying the Γ action and taking into consideration the long exact sequence of homology of the short exact sequence

$$0 \longrightarrow \mathrm{Div}_0(\Omega_{\mathcal{L}}) \longrightarrow \mathrm{Div}(\Omega_{\mathcal{L}}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

we get the connecting morphism

$$\begin{array}{ccc} \Gamma^{ab} \cong H_1(\Gamma, \mathbb{Z}) & \longrightarrow & H_0(\Gamma, \mathrm{Div}_0(\Omega_{\mathcal{L}})) \cong \mathrm{Div}_0(\Omega_{\mathcal{L}})_\Gamma \\ \gamma & \longrightarrow & \gamma p - p \end{array}$$

where $p \in \Omega_{\mathcal{L}}$ is any point.

Taking coinvariants of the integration map and composing with the one just above we get

$$\begin{array}{rcl} \Gamma^{ab} & \xrightarrow{\quad \times\!\!\!\!\int \cdot\, d \quad} & \mathrm{Hom}(\mathcal{M}(\mathcal{L}^*,\mathbb{Z})_0^\Gamma, K^*) \\ \gamma & \longmapsto & \left\{ \times\!\!\!\!\!\!\int_{\gamma p - p} d : \mu \mapsto \times\!\!\!\!\!\!\int_{\gamma p-p} d\mu \right\} \end{array}$$

Next, let us consider the graph $G_\Gamma := \mathcal{T}_K(\mathcal{L})/\Gamma$. It is finite and $\pi_\Gamma : \mathcal{T}_K(\mathcal{L}) \longrightarrow G_\Gamma$ is its universal covering, so we get an isomorphism $\Gamma^{ab} \cong H_1(G_\Gamma, \mathbb{Z})$ (we write z_γ the cycle corresponding to γ). Therefore, the natural pairing on the edges of the graph, which induces one on $H_1(G_\Gamma,\mathbb{Z})$, gives to us a symmetric definite positive form

$$(,)_\Gamma : \Gamma^{ab} \times \Gamma^{ab} \longrightarrow \mathbb{R}$$

Next note that an edge e of $\mathcal{T}_K(\mathcal{L})$ determines a divisor on $\Omega_\mathcal{L}$ by taking its boundary, that is its target minus its source: $\partial e = t(e) - s(e)$. Let us denote by $l(e)$ its length.

Lemma 2.

$$v_K\left(\times\!\!\!\!\!\!\int_{\partial e} d\mu \right) = l(e)\mu(e)$$

As a consequence we get the main result in order to prove that T/Λ is an abelian variety:

Theorem 3. *The map* $\mu : \Gamma^{ab} \longrightarrow \mathcal{M}(\mathcal{L}^*,\mathbb{Z})_0^\Gamma$ *defined by*

$$\mu_\gamma(e) := \frac{(\pi_\Gamma(e), z_\gamma)}{l(e)}$$

is an isomorphism and for all $\gamma, \gamma' \in \Gamma$, $\alpha \in \mathcal{T}_K(\mathcal{L})$ *it verifies*

$$(\gamma,\gamma')_\Gamma = v_K\left(\times\!\!\!\!\!\!\int_{\gamma\alpha-\alpha} d\mu_{\gamma'} \right)$$

Therefore, the integration map let us build a pairing

$$\begin{array}{rcl} \Gamma^{ab}\times\Gamma^{ab} & \xrightarrow{\quad \times\!\!\!\!\int (,) \quad} & K^* \\ (\gamma,\gamma') & \longmapsto & \times\!\!\!\!\!\!\int_{\gamma p-p} d\mu_{\gamma'} \end{array}$$

and taking its valuation we get:

$$v_K\left(\times\!\!\!\!\!\!\int (\gamma,\gamma') \right) = v_K\left(\times\!\!\!\!\!\!\int_{\gamma p-p} d\mu_{\gamma'} \right) = -\log\left| \times\!\!\!\!\!\!\int_{\gamma p-p} d\mu_{\gamma'} \right| =$$

$$= -\log\left| \times\!\!\!\!\!\!\int \right|_{\gamma\, \mathrm{r}_\mathcal{L}(p) - \mathrm{r}_\mathcal{L}(p)} d\mu_{\gamma'} = v_K\left(\times\!\!\!\!\!\!\int_{\gamma\, \mathrm{r}_\mathcal{L}(p)-\mathrm{r}_\mathcal{L}(p)} d\mu_{\gamma'} \right) = (\gamma,\gamma')_\Gamma$$

This, up to minor details, let us deduce that the pairing to K^* is symmetric and positive definite, and so that T/Λ is an abelian variety.

To show that it is the Jacobian of C_Γ, first, the integral map let us build a map

$$\mathrm{Div}_0(C_\Gamma)/\,\mathrm{Prin}(C_\Gamma) \longrightarrow T/\Lambda$$

Then, to see that it is an isomorphism we use three main tools:

A map $\tilde{\mu} : \mathcal{O}(\Omega_{\mathcal{L}})^* \longrightarrow \mathcal{M}(\mathcal{L}^*, \mathbb{Z})_0$ with $\mathrm{Ker}(\tilde{\mu}) = K^*$ and satisfying some nice conditions which can be enumerated under the ideas of "continuity", "naturality" (with respect to $\mathcal{L}$) and "equivariance" (with respect to Γ).

Actually, we have to say that this map is already used through the first part of the proof to get some of the (not so) "minor details".

The second big tool is the Poisson formula. Given an analytic function $u \in \mathcal{O}(\Omega_{\mathcal{L}})^*$ and points $z, z_0 \in \Omega_{\mathcal{L}}$ the next equality is satisfied:

$$\frac{u(z)}{u(z_0)} = \oint_{z-z_0} d\tilde{\mu}(u)$$

In third place, we have to deal explicitly with automorphic functions for Γ, that is, meromorphic functions f on $\Omega_{\mathcal{L}}$ such that there exists $c_f : \Gamma \longrightarrow K^*$ verifying $f(z) = c_f(\gamma) f(\gamma z)$ for all $z \in \Omega_{\mathcal{L}}$ and $\gamma \in \Gamma$.

We say explicitly because, although hidden, they are implicit in the first part of the proof (when proving that T/Λ is abelian). Indeed, fixed $z_0 \in \Omega_{\mathcal{L}}$ and $\mu \in \mathcal{M}(\mathcal{L}^*, \mathbb{Z})_0^{\Gamma}$, the function

$$f_\mu(z) := \oint_{z-z_0} d\mu$$

is Γ-automorphic.

These function satisfy a list of important properties:

– If we write $\mathcal{A}(\Omega_{\mathcal{L}})$ the group of automorphic functions for Γ, the map

$$\begin{array}{ccc} \mathcal{A}(\Omega_{\mathcal{L}}) & \longrightarrow & \mathrm{Hom}(\Gamma^{ab}, K^*) \\ f & \longmapsto & c_f \end{array}$$

is exhaustive.

– If f is automorphic and analytic, $\tilde{\mu}(f)$ is a Γ-invariant measure, and so is of the form μ_δ for some $\delta \in \Gamma$.

– Given $p, p' \in \Omega_{\mathcal{L}}$ we define the theta function associated to the divisor $p - p'$ by

$$\theta(p - p'; z) := \prod_{\gamma \in \Gamma} \frac{z - \gamma p}{z - \gamma p'}$$

Then, given $\delta \in \Gamma$ and $p \in \Omega_{\mathcal{L}}$ we have $\tilde{\mu}(\theta(p - \delta p; z)) = \mu_\delta$.

– We associate to a Γ-automorphic function f a finite degree zero divisor D_f^{Γ} such that the divisor of zeroes and poles of f in $\Omega_{\mathcal{L}}$ is

$$\sum_{\gamma \in \Gamma} \gamma \cdot D_f^{\Gamma}.$$

Therefore, by means of the isomorphism $\Gamma^{ab} \cong \mathcal{M}(\mathcal{L}^*, \mathbb{Z})_0^{\Gamma}$ we can see

$$\oint_{D_f^{\Gamma}} d \in \mathrm{Hom}(\Gamma^{ab}, K^*)/\Lambda$$

In this space we have $c_f = \oint_{D_f^{\Gamma}} d$.

References

[BR10] Matthew Baker and Robert Rumely. *Potential theory and dynamics on the Berkovich projective line*, volume 159 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[GvdP80] Lothar Gerritzen and Marius van der Put. *Schottky groups and Mumford curves*, volume 817 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[MD73] Yu. Manin and V. Drinfeld. Periods of p-adic Schottky groups. *J. Reine Angew. Math.*, 262/263:239–247, 1973. Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday.

[Mum72] David Mumford. An analytic construction of degenerating curves over complete local rings. *Compositio Math.*, 24(2):129–174, 1972.