Tate module tensor decompositions and Sato-Tate conjecture for varieties potentially of GL_2-type

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Outline

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Sato–Tate conjecture for elliptic curves

- E/k elliptic curve, $\mathfrak{p} \subset \mathcal{O}_k$ a prime of good reduction
  $\rightsquigarrow a_{\mathfrak{p}} = |\mathfrak{p}| + 1 - \#E(\mathcal{O}_k/\mathfrak{p})$

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Theorem (known cases of Sato–Tate)

If E has CM (Deuring, Hecke)
If k is a totally real field (2009, Barnet-Lamb, Geraghty, Harris, Taylor)
If k is a CM field (2018, 10 author's paper)

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- For non-generic A/k Serre defined:
  - $A \rightsquigarrow \mathrm{ST}_A \subset \mathrm{USp}_{2g}$ compact
  - $\mathfrak{p} \rightsquigarrow x_\mathfrak{p} \in \mathrm{Conj}(\mathrm{ST}_A)$ such that $\mathrm{charpoly}(x_\mathfrak{p}) = \overline{L}_\mathfrak{p}(T)$
  - Conjecture: $\{x_\mathfrak{p}\}_\mathfrak{p} \subset \mathrm{Conj}(\mathrm{ST}_A)$ equidist. w.r.t Haar measure of ST_A

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Strategy for proving Sato–Tate

- $\{x_\mathfrak{p}\}_\mathfrak{p} \subset \mathrm{Conj}(G)$ and $\rho\colon G \to \mathrm{GL}_d(\mathbb{C})$ representation

$$L(\rho, s) = \prod_{\mathfrak{p}} \det(1 - \rho(\tilde{x}_\mathfrak{p})|\mathfrak{p}|^{-s})^{-1} \text{ for } \mathrm{Re}(s) > 1$$

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If $\forall\ \rho \neq 1$ irred. $L(\rho, s)$ is invertible (analytic on $\mathrm{Re}(s) \geq 1$, $L(\rho, 1) \neq 0$) then Sato-Tate holds

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  - This is not how the known cases of Sato–Tate are proved
  - Potential modularity results are enough

Potential modularity and Sato–Tate

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(Allen-Calegari-Caraiani-Gee-Helm-Le Hung-Newton-Scholze-Taylor-Thorne)

Suppose V is the 2-dimensional ℓ-adic representations associated to an elliptic curve E/k with k a CM field (or to A/k of GL_2-type). There exists a finite Galois extension L/k such that $\mathrm{Sym}^m V_{G_L}$ is automorphic.

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  - L/L_i solvable $\rightsquigarrow V_{|L_i}$ automorphic
  - ψ_i is a Hecke character $\rightsquigarrow$ automorphic
  - Shahidi: $L(1, \pi_1 \times \pi_2) \neq 0$

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- Inspired by work of Noah Taylor who did it in dim $A = 2$

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- A similar result holds when $A_F \sim B^g$ and B an abelian surface with QM, an abelian surface with RM or an abelian fourfold with QM.

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- Happy birthday Jordi!

Tate module tensor decompositions and Sato-Tate conjecture for varieties potentially of GL_2-type

Francesc Fité (MIT) Xevi Guitart (UB)

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